Section 5.1  Trigonometric Functions of Real Numbers

In calculus and in the sciences many of the applications of the trigonometric functions require that the inputs be real numbers, rather than angles. By making this small but crucial change in our viewpoint, we can define the trigonometric functions in such a way that the inputs are real numbers. The definitions of the trig functions, and the identities that we have already met (and will meet later) will remain the same, and will be valid whether the inputs are angles or real numbers.

Here are some identities you need to know:

\[ \tan(t) = \frac{\sin(t)}{\cos(t)} \]
\[ \cot(t) = \frac{\cos(t)}{\sin(t)} \]

**Reciprocal Identities**

\[ \sec(t) = \frac{1}{\cos(t)}, \quad \cos(t) \neq 0 \]
\[ \csc(t) = \frac{1}{\sin(t)}, \quad \sin(t) \neq 0 \]
\[ \cot(t) = \frac{1}{\tan(t)}, \quad \tan(t) \neq 0 \]

**Pythagorean Identities**

\[ \sin^2(t) + \cos^2(t) = 1 \]
\[ 1 + \tan^2(t) = \sec^2(t) \]
\[ 1 + \cot^2(t) = \csc^2(t) \]

**Opposite Angle Identities**

\[ \sin(-t) = -\sin(t) \]
\[ \cos(-t) = \cos(t) \]
\[ \tan(-t) = -\tan(t) \]
\[ \cot(-t) = -\cot(t) \]
\[ \sec(-t) = \sec(t) \]
\[ \csc(-t) = -\csc(t) \]
Example 1: Given that $\cos(t) = -\frac{1}{4}$ and $\pi < t < \frac{3\pi}{2}$, find $\sin(t)$ and $\tan(t)$.

Example 2: If $\sin(t) = \frac{1}{4}$ and $\tan(t) < 0$, find $\sec(t)$.

Example 3: Use the opposite-angle identities to find:

a. $\sin\left(-\frac{2\pi}{3}\right)$

b. $\cos\left(-\frac{5\pi}{6}\right)$

c. $\tan\left(-\frac{\pi}{4}\right) + \cot\left(-\frac{3\pi}{4}\right)$
Here’s another set of identities:

**Periodicity**

The sine and cosine functions are periodic functions. That means that there is some number \( p \) such that \( f(x + p) = f(x) \). The number \( p \) is the **period** of the function. So

\[
\sin(t + 2\pi) = \sin(t) \quad \text{more generally} \quad \sin(t + 2k\pi) = \sin(t)
\]

\[
\cos(t + 2\pi) = \cos(t) \quad \text{more generally} \quad \cos(t + 2k\pi) = \cos(t)
\]

for all real numbers \( t \) and all integers \( k \).

The tangent and cotangent functions are also periodic functions. However, these functions repeat themselves when \( p = \pi \). So

\[
\tan(t + \pi) = \tan(t) \quad \text{more generally} \quad \tan(t + k\pi) = \tan(t)
\]

\[
\cot(t + \pi) = \cot(t) \quad \text{more generally} \quad \cot(t + k\pi) = \cot(t)
\]

for all real numbers \( t \) and all integers \( k \).

Note: the period for the sine and cosine functions is \( 2\pi \) while the period for the tangent and cotangent functions is \( \pi \).

The secant and cosecant functions are the reciprocal functions, so they will follow the same periodicity rules as sine and cosine.

\[
\sec(t + 2\pi k) = \sec(t)
\]

\[
\csc(t + 2\pi k) = \csc(t)
\]

for all real numbers \( t \) and all integers \( k \).

We will use the identities and periodicity to evaluate trig functions of real numbers.
Example 4: Evaluate $\tan\left(\frac{15\pi}{4}\right)$

Example 5: Evaluate $\cos\left(\frac{25\pi}{6}\right)$

Example 6: Evaluate $\sin\left(\frac{19\pi}{3}\right)$

Example 7: Evaluate $\sin\left(-\frac{20\pi}{3}\right)$
Example 8: Evaluate \[
\frac{\cos\left(\frac{19\pi}{3}\right)\tan\left(\frac{21\pi}{4}\right)}{\cos(8\pi)\sin\left(\frac{25\pi}{6}\right)}
\]

Example 9: Evaluate \[
\cot\left(\frac{21\pi}{4}\right) - \frac{\tan\left(\frac{17\pi}{4}\right)}{\cos(11\pi)\sin\left(\frac{17\pi}{6}\right)}
\]

Example 10: Simplify: \[
\cot(-t)\sec(-t)
\]