Section 6.2 – Double and Half Angle Formulas

Now suppose we are interested in finding \( \sin(2A) \). We can use the sum formula for sine to develop this identity:

\[
\sin(2A) = \sin(A + A) = \sin A \cos A + \sin A \cos A = 2 \sin A \cos A
\]

Similarly, we can develop a formula for \( \cos(2A) \):

\[
\cos(2A) = \cos(A + A) = \cos A \cos A - \sin A \sin A = \cos^2 A - \sin^2 A
\]

We can restate this formula in terms of sine only or in terms of cosine only by using the Pythagorean theorem and making a substitution. So we have:

\[
\cos(2A) = \cos^2 A - \sin^2 A = 1 - 2 \sin^2 A = 2 \cos^2 A - 1
\]

We can also develop a formula for \( \tan(2A) \):

\[
\tan(2A) = \tan(A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A} = \frac{2 \tan A}{1 - \tan^2 A}
\]

These three formulas are called the **double angle formulas for sine, cosine and tangent**.

Double – Angle Formulas

\[
\sin(2A) = 2 \sin A \cos A
\]
$\cos(2A) = \cos^2 A - \sin^2 A$  
(Also: $\cos(2A) = 2\cos^2 A - 1 = 1 - 2\sin^2 A$)

\[
\tan(2A) = \frac{2\tan A}{1 - \tan^2 A}
\]

Now we’ll look at the types of problems we can solve using these identities.

Example 1: Suppose that $\cos \alpha = -\frac{4}{7}$ and $\frac{\pi}{2} < \alpha < \pi$. Find

a. $\cos(2\alpha)$

b. $\sin(2\alpha)$

c. $\tan(2\alpha)$

Example 2: Simplify each:
a. \(2 \sin 45^\circ \cos 45^\circ\)

b. \(\cos^2 \frac{\pi}{9} - \sin^2 \frac{\pi}{9}\)

c. \(\frac{2 \tan 15^\circ}{1 - \tan^2 15^\circ}\)

d. \(1 - 2 \sin^2(6A)\)

Half – Angle Formulas
\[
\sin \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{2}}
\]

\[
\cos \frac{A}{2} = \pm \sqrt{\frac{1 + \cos A}{2}}
\]

\[
\tan \frac{A}{2} = \frac{\sin A}{1 + \cos A} = \frac{1 - \cos A}{\sin A}
\]

**Note:** In the half-angle formulas the ± symbol is intended to mean either positive or negative but not both, and the sign before the radical is determined by the quadrant in which the angle \( \frac{A}{2} \) terminates.

Now we’ll look at the kinds of problems we can solve using half-angle formulas.

**Example 3:** Use a half-angle formula to find the exact value of each.

a. \( \sin 15^\circ \)

b. \( \cos \left( \frac{5\pi}{8} \right) \)

c. \( \tan \left( \frac{7\pi}{12} \right) \)

**Example 4:** Answer these questions for \( \cos \theta = \frac{4}{9}, \frac{3\pi}{2} < \theta < 2\pi \).

a. In which quadrant does the terminal side of the angle lie?
b. Complete the following: \[ \_ < \frac{\theta}{2} < \_ \]

c. In which quadrant does the terminal side of \( \frac{\theta}{2} \) lie?

d. Determine the sign of \( \sin \frac{\theta}{2} \).

e. Determine the sign of \( \cos \frac{\theta}{2} \).

f. Find the exact value of \( \sin \frac{\theta}{2} \).

g. Find the exact value of \( \cos \frac{\theta}{2} \).

h. Find the exact value of \( \tan \frac{\theta}{2} \).