Section 8.3  Hyperbolas

**Definition:** A *hyperbola* is the set of all points, the difference of whose distances from two fixed points is constant. Each fixed point is called a *focus* (plural = *foci*).

The *focal axis* is the line passing through the foci.

**Basic “vertical” hyperbola:**

- **Equation:** \( \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \)
- **Asymptotes:** \( y = \pm \frac{a}{b} x \)
- **Foci:** \( (0, \pm c) \), where \( c^2 = a^2 + b^2 \)
- **Vertices:** \( (0, \pm a) \)
- **Eccentricity:** \( \frac{c}{a} \) (\( > 1 \))

**Basic “horizontal” hyperbola:**

- **Equation:** \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \)
- **Asymptotes:** \( y = \pm \frac{b}{a} x \)
- **Foci:** \( (\pm c, 0) \), where \( c^2 = a^2 + b^2 \)
- **Vertices:** \( (\pm a, 0) \)
- **Eccentricity:** \( \frac{c}{a} \) (\( > 1 \))
**Note:** The **transverse axis** is the line segment joining the two vertices. The **conjugate axis** is the line segment perpendicular to the transverse axis, passing through the center and extending a distance \(b\) on either side of the center. (These terms will make more sense after we do the graphing examples.)

The **conjugate axis** of the hyperbola is the line segment through the center of the hyperbola and perpendicular to the transverse axis with endpoints \((0,-b)\) and \((0, b)\).

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1
\]

**Center:** \((0,0)\)

**Foci:** \((-c, 0)\) and \((c, 0)\), where \(c^2 = a^2 + b^2\)

**Vertices:** \(V_1(-a,0)\) and \(V_2(a,0)\)

**Transverse Axis:** \(V_1V_2\)  \hspace{1cm} \text{Length of Transverse Axis:} \ 2a

**Conjugate Axis:** \(AB\)  \hspace{1cm} \text{Length of Conjugate Axis:} \ 2b

The **eccentricity** of a hyperbola is given by the formula \(e = \frac{c}{a}\).

The lines \(y = \frac{b}{a}x\) and \(y = -\frac{b}{a}x\) are **slant asymptotes** for the hyperbola since it can be shown that as \(|x|\) becomes large, \(y \to \pm\frac{b}{a}x\).
Graphing hyperbolas:

To graph a hyperbola with center at the origin:

- Rearrange into the form \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \) or \( \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \).

- Decide if it’s a “horizontal” or “vertical” hyperbola.
  - If \( x^2 \) comes first, it’s horizontal (vertices are on \( x \)-axis).
  - If \( y^2 \) comes first, it’s vertical (vertices are on \( y \)-axis).

- Use the square root of the number under \( x^2 \) to determine how far to measure in \( x \)-direction.

- Use the square root of the number under \( y^2 \) to determine how far to measure in \( y \)-direction.

- Draw a box with these measurements.

- Draw diagonals through the box. These are the asymptotes. Use the dimensions of the box to determine the slope and write the equations of the asymptotes.

- Put the vertices at the edge of the box on the correct axis. Then draw a hyperbola, making sure it approaches the asymptotes smoothly.

- \( c^2 = a^2 + b^2 \) where \( a^2 \) and \( b^2 \) are the denominators.

- The foci are located \( c \) units from the center, on the same axis as the vertices.

When graphing hyperbolas, you will need to find the orientation, center, values for \( a, b \) and \( c \), lengths of transverse and converse axes, vertices, foci, equations of the asymptotes, and eccentricity.
Example 1: Find all relevant information and graph $\frac{x^2}{36} - \frac{y^2}{4} = 1$.

Vertices:
Foci:
Eccentricity:
Transverse Axis:
Length of transverse axis:
Conjugate axis:
Length of conjugate axis:
Slant Asymptotes:
**Example 2:** Find all relevant information and graph \( \frac{y^2}{4} - \frac{x^2}{9} = 1 \).

**Vertices:**

**Foci:**

**Eccentricity:**

**Transverse Axis:**

Length of transverse axis:

**Conjugate axis:**

Length of conjugate axis:

**Slant Asymptotes:**
The equation of a hyperbola with center not at the origin:  
Center: (h, k)

\[
\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \text{ or } \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1
\]

To graph a hyperbola with center not at the origin:

- Rearrange (complete the square if necessary) to look like

\[
\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \text{ or } \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1.
\]

- Start at the center \((h, k)\) and then graph it as before.

- To write down the equations of the asymptotes, start with the equations of the asymptotes for the similar hyperbola with center at the origin. Then replace \(x\) with \(x-h\) and replace \(y\) with \(y-k\).

Example 3:  Write the equation in standard form, find all relevant information and graph

\[9x^2 - 16y^2 - 18x + 96y = 279.\]
Example 4: Write an equation of the hyperbola with center at (-2, 3), one vertex is at (-2, -2) and eccentricity is 2.

Example 5: Write an equation of the hyperbola if the vertices are (4, 0) and (4, 8) and the asymptotes have slopes ±1.