Multiparameter Waveform Inversion via Reduced Order Modeling

Alexander V. Mamonov[1],
Liliana Borcea[2], Josselin Garnier[3] and Jörn Zimmerling[4]

[1]University of Houston,
[2]Columbia University, [3]Ecole Polytechnique, [4]Uppsala University

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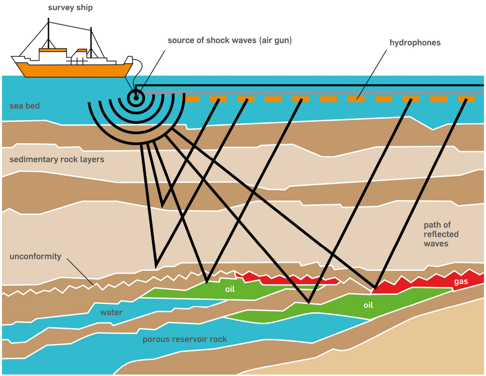
Motivation: seismic exploration



- **Reduced order model** (ROM) framework for acoustic parameter estimation:
  1. Construct a **data-driven** ROM from the data
  2. Formulate **velocity and density** estimation as **ROM misfit** minimization
- ROM misfit objective is much better behaved than conventional FWI least squares data misfit objective

First order forward model

- **Setting**: array of m sources/receivers sends and measures waves modeled by the first-order system

$$\begin{aligned} &\partial_t v^s(t,\mathbf{x}) + \rho^{-1}(\mathbf{x})\nabla p^s(t,\mathbf{x}) = f(t)\sigma_s(\mathbf{x}), && s = 1,\ldots,m, \\ &\partial_t p^s(t,\mathbf{x}) + K(\mathbf{x})\nabla\cdot v^s(t,\mathbf{x}) = 0, && \mathbf{x}\in\Omega, \end{aligned}$$

  with quiescent initial conditions

$$p^s(t,\mathbf{x}) = 0, \quad v^s(t,\mathbf{x}) = 0, \quad t \ll 0,$$

  and compactly supported source pulse

$$\mathrm{supp}(f) \subseteq [-t_f, t_f]$$

- Sources/receivers are collocated, s can also index different **polarizations** at the same physical location

Measurements and inverse problem

- Measurements form a time-dependent matrix $\mathbf{F} \in \mathbb{R}^{m \times m}$ with entries

$$\mathbf{F}_{sr}(t) = \int_{\Omega} \sigma_r(\mathbf{x}) \cdot v^s(t, \mathbf{x}) d\mathbf{x}, \quad s, r = 1, \ldots, m$$

- Acoustic velocity is

$$c(\mathbf{x}) = \sqrt{\frac{K(\mathbf{x})}{\rho(\mathbf{x})}}$$

- **Multiparameter waveform inversion (MWI)** problem: estimate velocity $c(\mathbf{x})$ and density $\rho(\mathbf{x})$ in Ω from the measurements $\mathbf{F}(t)$
- Velocity-density formulation is known to be better for MWI to minimize cross-talk

Symmetrized forward model

- Change of coordinates

$$\phi^s(t,\mathbf{x}) = \begin{bmatrix} \sqrt{\rho(\mathbf{x})}\, v^s(t+t_f,\mathbf{x}) \\ \frac{1}{\sqrt{K(\mathbf{x})}} p^s(t+t_f,\mathbf{x}) \end{bmatrix}$$

transforms the forward problem to

$$\partial_t \phi^s + A\phi^s = 0, \quad s = 1,\ldots,m,$$

with appropriately chosen initial condition $\phi^s(0,\mathbf{x})$ and

$$A = \begin{bmatrix} 0 & \frac{1}{\sqrt{\rho(\mathbf{x})}} \mathrm{grad} \left[\sqrt{K(\mathbf{x})} \cdot \right] \\ \sqrt{K(\mathbf{x})} \mathrm{div} \left[\frac{1}{\sqrt{\rho(\mathbf{x})}} \right] & 0 \end{bmatrix}$$

is **skew-symmetric**

Snapshots and projection ROM

- Assemble wavefields for all sources into

$$\phi(t,\mathbf{x}) = \left[\phi^1(t,\mathbf{x}), \ldots, \phi^m(t,\mathbf{x})\right]$$

- Fix discrete times $t_k = k\tau$, $k = 0, 1, \ldots, n-1$ and define **snapshots**

$$\phi_k(\mathbf{x}) = \phi(t_k, \mathbf{x})$$

  and also

$$\boldsymbol{\Phi}(\mathbf{x}) = \left[\phi_0(\mathbf{x}), \phi_1(\mathbf{x}), \ldots, \phi_{n-1}(\mathbf{x})\right] : \ \Omega \to \mathbb{R}^{1\times mn}$$

- We want to construct a **reduced order model (ROM)** by projecting

$$\partial_t \phi + A\phi = \mathbf{0}$$

  onto

$$\mathrm{span}(\boldsymbol{\Phi})$$

Mass and stiffness matrices

- Projection ROM is defined by **mass** matrix

$$\mathbf{M} = \int_\Omega \mathbf{\Phi}(\mathbf{x})^T \mathbf{\Phi}(\mathbf{x}) d\mathbf{x} \in \mathbb{R}^{mn \times mn},$$

  and **stiffness** matrix

$$\mathbf{S} = \int_\Omega \mathbf{\Phi}(\mathbf{x})^T [A\mathbf{\Phi}(\mathbf{x})] d\mathbf{x} \in \mathbb{R}^{mn \times mn}$$

- Problem: both matrices require **internal data**: wavefield snapshots $\mathbf{\Phi}(\mathbf{x})$ for $\mathbf{x} \in \Omega$
- Can we get them from **sensor data**: $\mathbf{F}(t)$?

Data-driven ROM

- Convert measurements $\mathbf{F}(t)$ to

$$\mathbf{D}_k = \int_\Omega \phi_0^T(\mathbf{x})\phi_k(\mathbf{x})d\mathbf{x} = \int_{-t_f}^{t_f} f(t)\mathbf{F}(t+t_k)dt, \quad k = 0,\ldots,n-1$$

- Snapshots satisfy

$$\phi_k(\mathbf{x}) = e^{-t_k A}\phi_0(\mathbf{x}),$$

so we have **data-driven** formula for **mass matrix blocks**

$$\begin{aligned}\mathbf{M}_{jk} &= \int_\Omega \phi_j^T(\mathbf{x})\phi_k(\mathbf{x})d\mathbf{x} \\ &= \int_\Omega \phi_0^T(\mathbf{x})e^{t_j A}e^{-t_k A}\phi_0(\mathbf{x})d\mathbf{x} \\ &= \int_\Omega \phi_0^T(\mathbf{x}) \underbrace{\left[e^{-(t_k-t_j)A}\phi_0(\mathbf{x})\right]}_{\phi_{k-j}(\mathbf{x})} d\mathbf{x} = \mathbf{D}_{k-j} \in \mathbb{R}^{m\times m}\end{aligned}$$

ROM for waveform inversion

- Previously, we used both mass and stiffness matrices to solve FWI
- Recently, we discovered that using $\mathbf{M}$ alone might be sufficient
- Consider **block-Cholesky** factorization of the mass matrix

$$\mathbf{M} = \mathbf{R}^T \mathbf{R}$$

corresponding to **block Gram-Schmidt** of the snapshots

$$\mathbf{\Phi}(\mathbf{x}) = \mathbf{Q}(\mathbf{x})\mathbf{R}$$

- Snapshots $\mathbf{\Phi}(\mathbf{x})$ for $\mathbf{x} \in \Omega$ are inaccessible, and so are $\mathbf{Q}(\mathbf{x})$
- Take a guess for the medium $\widetilde{c}$, $\widetilde{\rho}$ and approximate

$$\mathbf{\Phi}(\mathbf{x}) \approx \widetilde{\mathbf{\Phi}}(\mathbf{x}) = \mathbf{Q}(\mathbf{x}; \widetilde{c}, \widetilde{\rho})\mathbf{R}$$

- Approximation $\widetilde{\mathbf{\Phi}}(\mathbf{x})$ **fits the data**, since

$$\int_\Omega \widetilde{\mathbf{\Phi}}^T(\mathbf{x})\widetilde{\mathbf{\Phi}}(\mathbf{x}) d\mathbf{x} = \mathbf{R}^T\mathbf{R} = \mathbf{M},$$

but **does not satisfy wave equation**

Improved ROM with enriched projection subspace

- The ROM can be improved by enriching the projection subspace span($\boldsymbol{\Phi}$)
- If
$$\phi_k(\mathbf{x}) = \begin{bmatrix} \phi_k^v(\mathbf{x}) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \phi_k^p(\mathbf{x}) \end{bmatrix},$$
we can compute in a data-driven manner the mass matrices for velocity and pressure snapshots separately
$$\mathbf{M}_{jk}^v = \int_\Omega \left[\phi_j^v(\mathbf{x})\right]^T \phi_k^p(\mathbf{x}) d\mathbf{x}$$
$$\mathbf{M}_{jk}^p = \int_\Omega \left[\phi_j^v(\mathbf{x})\right]^T \phi_k^p(\mathbf{x}) d\mathbf{x}$$
- The corresponding block-Cholesky factor then takes the form
$$\widehat{\mathbf{R}} = \begin{bmatrix} \mathbf{R}^v & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^p \end{bmatrix}$$

ROM for multiparameter waveform inversion

- **Enforce wave equation** by minimizing

$$\underset{\widetilde{c},\widetilde{\rho}}{\text{minimize}} \left\| \mathbf{Q}(\mathbf{x};\widetilde{c},\widetilde{\rho})\mathbf{R}(\widetilde{c},\widetilde{\rho}) - \mathbf{Q}(\mathbf{x};\widetilde{c},\widetilde{\rho})\mathbf{R} \right\|_F^2$$

  or simply

$$\underset{\widetilde{c},\widetilde{\rho}}{\text{minimize}} \left\| \mathbf{R}(\widetilde{c},\widetilde{\rho}) - \mathbf{R} \right\|_F^2$$

- In practice, what works even better is

$$\underset{\widetilde{c},\widetilde{\rho}}{\text{minimize}} \left\| \mathbf{R}(\widetilde{c},\widetilde{\rho})\mathbf{R}^{-1} - \mathbf{I}_{mn} \right\|_F^2$$

- Compare to **conventional MFWI** optimization

$$\underset{\widetilde{c},\widetilde{\rho}}{\text{minimize}} \int_0^{t_{\max}} \left\| \mathbf{F}(t;\widetilde{c},\widetilde{\rho}) - \mathbf{F}(t) \right\|_F^2 dt,$$

  prone to **cycle skipping** (local minima), slow convergence, need a good initial guess, etc.

Numerical experiments

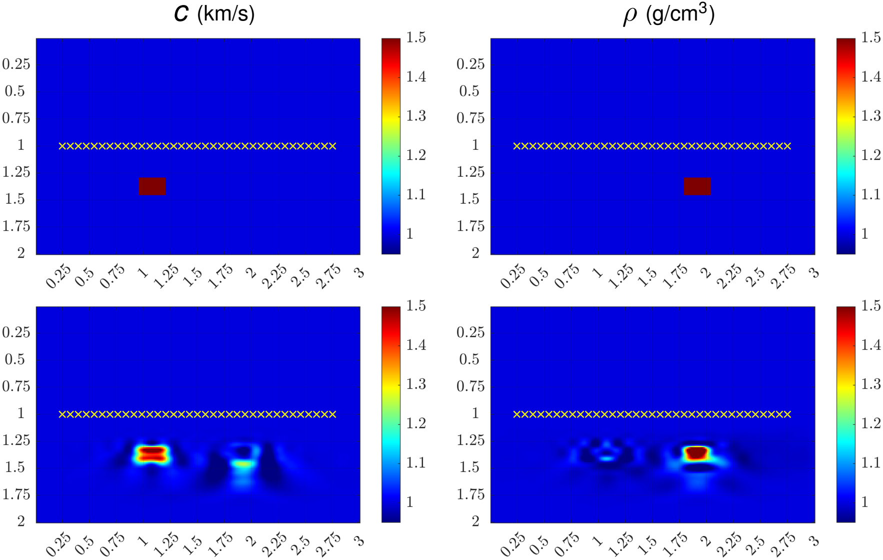
- Band-limited source wavelet

$$f(t) = \frac{\cos(\omega_0 t)}{\sqrt{2\pi} B_\omega} e^{-\frac{(B_\omega t)^2}{2}},$$

  with **central frequency** $\omega_0 = 2\pi(6Hz)$ and **bandwidth** $B_\omega = 2\pi(4Hz)$

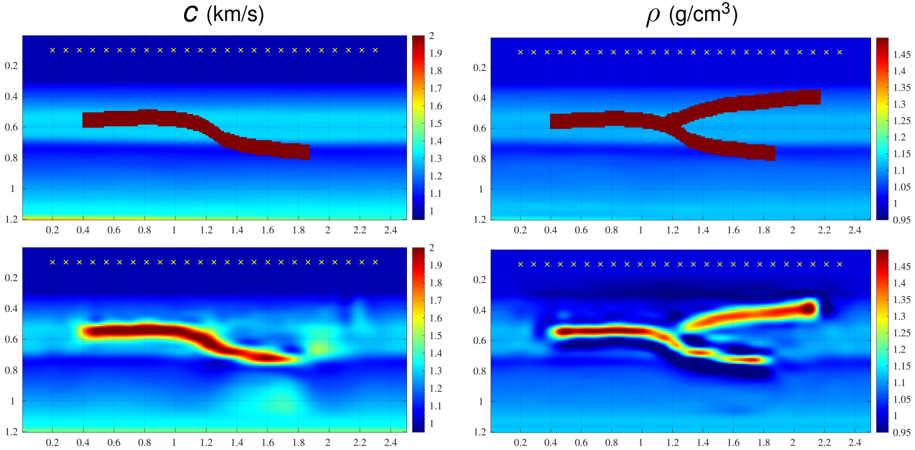
- Two **polarizations** (vertical and horizontal) per each sensor
- ROM based MWI is solved via **Gauss-Newton** iteration regularized with **adaptive Tikhonov regularization**
- Three numerical examples:
  1. Two inclusions to assess **cross-talk**
  2. Synthetic crack model
  3. Section of two-parameter **Marmousi** model

Numerical results: two inclusions



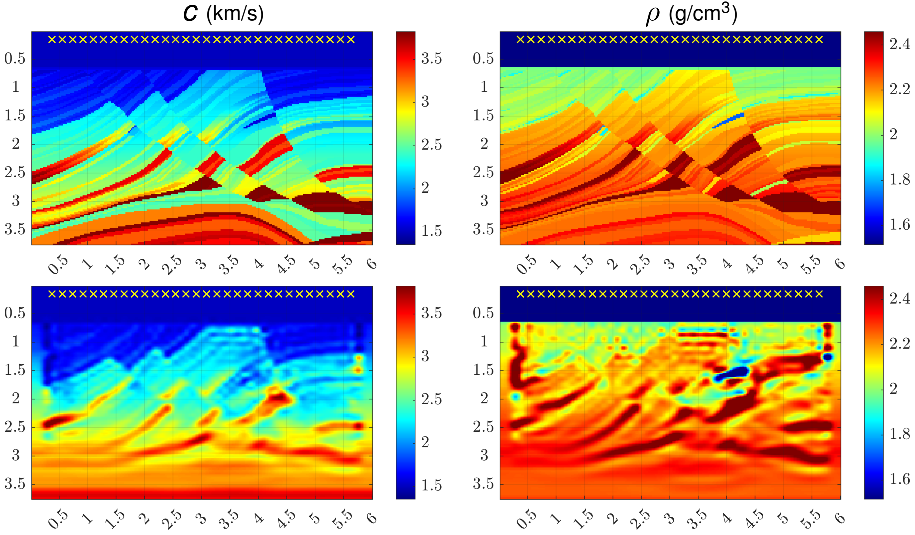
$m = 2 \cdot 35 = 70$ sources/receivers, 15 iterations

Numerical results: crack



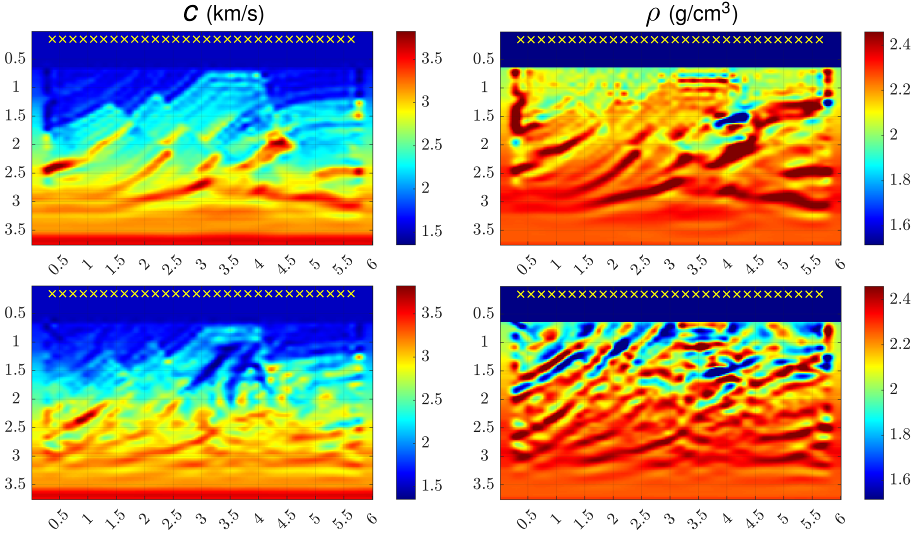
$m = 2 \cdot 25 = 50$ sources/receivers, 40 iterations

Numerical results: two-parameter Marmousi model



$m = 2 \cdot 30 = 60$ sources/receivers, 50 iterations

Numerical results: two-parameter Marmousi model



Comparison of ROM-based approach (top row) to conventional MFWI (bottom row).

Conclusions and future work

- We introduced **ROM** framework for multiparameter waveform inversion (MWI)
- **Time domain** formulation is essential, linear algebraic analogues of **causality**: Gram-Schmidt, Cholesky
- Separate MWI into **two steps**:
  1. Construct the ROM from **data**
  2. Use **ROM misfit** as optimization objective
- Much better behaved than conventional FWI least squares data misfit even for **band-limited** sources: ROM misfit optimization objective is very close to **convex**
- **Robust** version exists for **noisy** and/or **incomplete data**, requires non-trivial regularization of ROM construction process

Future work:

- Extend to **vectorial** problems, e.g., electromagnetics, elasticity

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