Resistor Networks and Optimal Grids for Electrical Impedance Tomography with Partial Boundary Measurements

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Electrical Impedance Tomography

1. EIT with resistor networks and optimal grids
2. Conformal and quasi-conformal mappings
3. Pyramidal networks and sensitivity grids
4. Two-sided problem and networks
5. Numerical results
6. Conclusions
Electrical Impedance Tomography: Physical problem

- Physical problem: determine the electrical conductivity inside an object from the simultaneous measurements of voltages and currents on (a part of) its boundary

- Applications:
  - Original: geophysical prospection
  - More recent: medical imaging

- Both cases in practice have measurements restricted to a part of object’s boundary
Partial data EIT: mathematical formulation

- Two-dimensional problem $\Omega \subset \mathbb{R}^2$
- Equation for electric potential $u$
  \[ \nabla \cdot (\sigma \nabla u) = 0, \quad \text{in } \Omega \]
- Dirichlet data $u|_B = \phi \in H^{1/2}(B)$ on $B = \partial \Omega$
- Dirichlet-to-Neumann (DtN) map $\Lambda_\sigma : H^{1/2}(B) \to H^{-1/2}(B)$
  \[ \Lambda_\sigma \phi = \sigma \frac{\partial u}{\partial \nu} \bigg|_B \]

Partial data case:
- Split the boundary $B = B_A \cup B_I$, accessible $B_A$, inaccessible $B_I$
- Dirichlet data: $\text{supp } \phi_A \subset B_A$
- Measured current flux: $J_A = (\Lambda_\sigma \phi_A)|_{B_A}$
- Partial data EIT: find $\sigma$ given all pairs $(\phi_A, J_A)$
Existence, uniqueness and stability

Existence and uniqueness:

- Full data: solved completely for any positive $\sigma \in L^\infty(\Omega)$ in 2D (Astala, Päivärinta, 2006)
- Partial data: for $\sigma \in C^{4+\alpha}(\overline{\Omega})$ and an arbitrary open $B_A$ (Imanuvilov, Uhlmann, Yamamoto, 2010)

Stability (full data):

- For $\sigma \in L^\infty(\Omega)$ the problem is unstable (Alessandrini, 1988)
- Logarithmic stability estimates (Barcelo, Faraco, Ruiz, 2007) under certain regularity assumptions
  \[
  \|\sigma_1 - \sigma_2\|_\infty \leq C \|\log \|\Lambda_{\sigma_1} - \Lambda_{\sigma_2}\|_{H^1/2(B) \to H^{-1/2}(B)}\|^{-a}
  \]
  The estimate is sharp (Mandache, 2001), additional regularity of $\sigma$ does not help
- Exponential ill-conditioning of the discretized problem
- Resolution is severely limited by the noise, regularization is required
Numerical methods for EIT

1. **Linearization**: Calderon’s method, one-step Newton, backprojection.
2. **Optimization**: typically output least squares with regularization.
3. **Layer peeling**: find $\sigma$ close to $B$, peel the layer, update $\Lambda_\sigma$, repeat.
4. **D-bar method**: non-trivial implementation.

**Resistor networks and optimal grids**
- Uses the close connection between the continuum inverse problem and its discrete analogue for resistor networks
- Fit the measured continuum data exactly with a resistor network
- Interpret the resistances as averages over a special (optimal) grid
- Compute the grid once for a known conductivity (constant)
- Optimal grid depends weakly on the conductivity, grid for constant conductivity can be used for a wide range of conductivities
- Obtain a pointwise reconstruction on an optimal grid
- Use the network and the optimal grid as a non-linear preconditioner to improve the reconstruction using a single step of traditional (regularized) Gauss-Newton iteration
Finite volume discretization and resistor networks

Finite volume discretization, staggered grid

Kirchhoff matrix

\[ K = A \text{diag}(\gamma) A^T \succeq 0 \]

Interior \( I \), boundary \( B \), \( |B| = n \)

Potential \( u \) is \( \gamma \)-harmonic

\[ K_I \cdot u = 0, \ u_B = \phi \]

Discrete DtN map \( \Lambda_{\gamma} \in \mathbb{R}^{n \times n} \)

Schur complement:

\[ \Lambda_{\gamma} = K_{BB} - K_{Bl} K_{ll}^{-1} K_{IB} \]

Discrete inverse problem:

Knowing \( \Lambda_{\gamma}, A \), find \( \gamma \)

What network topologies are good?
Discrete inverse problem: circular planar graphs

- Circular pair \((P; Q), P \subset B, Q \subset B\)
- \(\pi(\Gamma)\) all \((P; Q)\) connected through \(\Gamma\) by disjoint paths
- **Critical** \(\Gamma\): removal of any edge breaks some connection in \(\pi(\Gamma)\)
- Uniquely recoverable from \(\Lambda\) iff \(\Gamma\) is critical (Curtis, Ingerman, Morrow, 1998)
- Characterization of DtN maps of critical networks \(\Lambda_\gamma\)
  - Symmetry \(\Lambda_\gamma = \Lambda_\gamma^T\)
  - Conservation of current \(\Lambda_\gamma \mathbf{1} = \mathbf{0}\)
  - Total non-positivity \(\det[-\Lambda_\gamma(P; Q)] \geq 0\)

Planar graph \(\Gamma\)
- \(I\) embedded in the unit disk \(\mathbb{D}\)
- \(B\) in cyclic order on \(\partial \mathbb{D}\)
Discrete vs. continuum

- Measurement (electrode) functions $\chi_j$, $\text{supp} \chi_j \subset B_A$
- Measurement matrix $\mathcal{M}_n(\Lambda_\sigma) \in \mathbb{R}^{n \times n}$: $[\mathcal{M}_n(\Lambda_\sigma)]_{i,j} = \int_B \chi_i \Lambda_\sigma \chi_j dS$, $i \neq j$
- $\mathcal{M}_n(\Lambda_\sigma)$ has the properties of a DtN map of a resistor network (Morrow, Ingerman, 1998)
- How to interpret $\gamma$ obtained from $\Lambda_\gamma = \mathcal{M}_n(\Lambda_\sigma)$?
- From finite volumes define the reconstruction mapping

$$Q_n [\Lambda_\gamma] : \sigma^*(P_{\alpha,\beta}) = \frac{\gamma_{\alpha,\beta}}{\gamma_{\alpha,\beta}^{(1)}}, \text{ piecewise linear interpolation away from } P_{\alpha,\beta}$$

- Optimal grid nodes $P_{\alpha,\beta}$ are obtained from $\gamma_{\alpha,\beta}^{(1)}$, a solution of the discrete problem for constant conductivity $\Lambda_\gamma^{(1)} = \mathcal{M}_n(\Lambda_1)$.
- The reconstruction is improved using a single step of preconditioned Gauss-Newton iteration with an initial guess $\sigma^*$

$$\min_\sigma \| Q_n [\mathcal{M}_n(\Lambda_\sigma)] - \sigma^* \|$$
Optimal grids in the unit disk: full data

- Tensor product grids uniform in $\theta$, adaptive in $r$
- Layered conductivity $\sigma = \sigma(r)$
- Admittance $\Lambda_\sigma e^{ik\theta} = R(k) e^{ik\theta}$
- For $\sigma \equiv 1$ $R(k) = |k|$
  $\Lambda_1 = \sqrt{-\frac{\partial^2}{\partial \theta^2}}$
- Discrete analogue $M_n(\Lambda_1) = \sqrt{\text{circ}(-1, 2, -1)}$

- Discrete admittance $R_n(\lambda) = \frac{1}{\gamma_1 + \frac{1}{\gamma_2 \lambda^2 + \ldots + \frac{1}{\gamma_{m+1} \lambda^2 + \gamma_{m+1}}}}$

- Rational interpolation
  $R(k) = \frac{k}{\omega_k^{(n)} R_n(\omega_k^{(n)})}$

- Optimal grid $R_{n}^{(1)}(\omega_k^{(n)}) = \omega_k^{(n)}$

- Closed form solution available (Biesel, Ingerman, Morrow, Shore, 2008)

- Vandermonde-like system, exponential ill-conditioning
Transformation of the EIT under diffeomorphisms

- Optimal grids were used successfully to solve the full data EIT in $\mathbb{D}$
- Can we reduce the partial data problem to the full data case?
- Conductivity under diffeomorphisms $G$ of $\Omega$: **push forward** $\tilde{\sigma} = G_*(\sigma)$, $\tilde{u}(x) = u(G^{-1}(x))$

$$\tilde{\sigma}(x) = \left. \frac{G'(y)\sigma(y)(G'(y))^T}{|\det G'(y)|} \right|_{y=G^{-1}(x)}$$

- Matrix valued $\tilde{\sigma}(x)$, anisotropy!
- Anisotropic EIT is not uniquely solvable
- Push forward for the DtN: $(g_*\Lambda_\sigma)\phi = \Lambda_\sigma(\phi \circ g)$, where $g = G|_B$
- Invariance of the DtN: $g_*\Lambda_\sigma = \Lambda_{G_*\sigma}$
- Push forward, solve the EIT for $g_*\Lambda_\sigma$, pull back
- Must preserve isotropy, $G'(y)(G'(y))^T = I \Rightarrow \text{conformal } G$
- Conformal automorphisms of the unit disk are Möbius transforms
Conformal and quasi-conformal mappings

Conformal automorphisms of the unit disk

\[ \beta = \tau \frac{n+1}{2} \]

\[ -\beta = \tau \frac{n+3}{2} \]

\[ \alpha = \theta \frac{n+1}{2} \]

\[ -\alpha = \theta \frac{n+3}{2} \]

\[ F : \theta \rightarrow \tau, \ G : \tau \rightarrow \theta. \] Primary \times, dual \circ, \ n = 13, \ \beta = 3\pi/4. \]

Positions of point-like electrodes prescribed by the mapping.
Conformal and quasi-conformal mappings

Conformal mapping grids: limiting behavior

- No conformal limiting mapping
- Single pole moves towards $\partial \mathbb{D}$ as $n \to \infty$
- Accumulation around $\tau = 0$
- No asymptotic refinement in angle as $n \to \infty$
- Hopeless?
- Resolution bounded by the instability, $n \to \infty$ practically unachievable

Primary $\times$, dual $\circ$, limits $\nabla$,

$n = 37, \beta = 3\pi/4.$
Quasi-conformal mappings

- Conformal $w$, Cauchy-Riemann: $\frac{\partial w}{\partial z} = 0$, how to relax?
- Quasi-conformal $w$, Beltrami: $\frac{\partial w}{\partial z} = \mu(z) \frac{\partial w}{\partial z}$
- Push forward $w_*(\sigma)$ is no longer isotropic
- Anisotropy of $\tilde{\sigma} \in \mathbb{R}^{2 \times 2}$ is $\kappa(\tilde{\sigma}, z) = \frac{\sqrt{L(z)} - 1}{\sqrt{L(z)} + 1}$, $L(z) = \frac{\lambda_1(z)}{\lambda_2(z)}$

Lemma

Anisotropy of the push forward is given by $\kappa(w_*(\sigma), z) = |\mu(z)|$.

- Mappings with fixed values at $B$ and $\min \|\mu\|_{\infty}$ are extremal
- Extremal mappings are Teichmüller (Strebel, 1972)

$$\mu(z) = \|\mu\|_{\infty} \frac{\phi(z)}{|\phi(z)|}, \phi \text{ holomorphic in } \Omega$$
Computing the extremal quasi-conformal mappings

- Polygonal Teichmüller mappings
- Polygon is a unit disk with $N$ marked points on the boundary circle
- Can be decomposed as
  \[ W = \Psi^{-1} \circ A_K \circ \Phi, \]
  where $\Psi = \int \sqrt{\psi(z)} \, dz$, $\Phi = \int \sqrt{\phi(z)} \, dz$, $A_K$ - constant affine stretching
- $\phi, \psi$ are rational with poles and zeros of order one on $\partial \mathbb{D}$
- Recall Schwarz-Christoffel
  \[ s(z) = a + b \int \prod_{k=1}^{N} \left( 1 - \frac{\zeta}{z_k} \right)^{\alpha_k-1} \, d\zeta \]
- $\Psi, \Phi$ are Schwarz-Christoffel mappings to rectangular polygons
Conformal and quasi-conformal mappings

Polygonal Teichmüller mapping: the grids

The optimal grid with $n = 15$ under the Teichmüller mappings. Left: $K = 0.8$; right: $K = 0.66$. 
EIT with pyramidal networks: motivation

- Pyramidal (standard) graphs $\Sigma_n$
- Topology of a network accounts for the inaccessible boundary
- Criticality and reconstruction algorithm proved for pyramidal networks
- How to obtain the grids?
- Grids have to be purely 2D (no tensor product)
- Use the sensitivity analysis (discrete an continuum) to obtain the grids
- General approach works for any simply connected domain
Pyramidal network \((\Sigma_n, \gamma)\), \(n = 2m\) is uniquely recoverable from its DtN map \(\Lambda^{(n)}\) using the layer peeling algorithm. Conductances are computed with

\[
\gamma(e_{p,h}) = \left(\Lambda_{p,E(p,h)} + \Lambda_{p,C} \Lambda_{Z,C}^{-1} \Lambda_{Z,E(p,h)}\right) 1_{E(p,h)},
\]

\[
\gamma(e_{p,v}) = \left(\Lambda_{p,E(p,v)} + \Lambda_{p,C} \Lambda_{Z,C}^{-1} \Lambda_{Z,E(p,v)}\right) 1_{E(p,v)}.
\]

The DtN map is updated using

\[
\Lambda^{(n-2)} = -K_S - K_{SB} P^T \left(P \left(\Lambda^{(n)} - K_{BB}\right) P^T\right)^{-1} P K_{BS}.
\]

The formulas are applied recursively to \(\Sigma_n, \Sigma_{n-2}, \ldots, \Sigma_2\).
Sensitivity grids: motivation
**Sensitivity grids**

- Proposed by F. Guevara Vasquez
- Sensitivity functions

\[
\frac{\delta \gamma_{\alpha,\beta}}{\delta \sigma} = \left[ \left( \frac{\partial \Lambda_{\gamma}}{\partial \gamma} \right)^{-1} \mathcal{M}_n \left( \frac{\delta \Lambda_{\sigma}}{\delta \sigma} \right) \right]_{\alpha,\beta}
\]

where \( \Lambda_{\gamma} = \mathcal{M}_n(\Lambda_{\sigma}) \)

- The optimal grid nodes \( P_{\alpha,\beta} \) are roughly

\[
P_{\alpha,\beta} \approx \arg \max_{x \in \Omega} \frac{\delta \gamma_{\alpha,\beta}}{\delta \sigma}(x)
\]

- Works for any domain and any network topology!

Sensitivity grid, \( n = 16 \).
Two-sided problem: $B_A$ consists of two disjoint segments of the boundary. Example: cross-well measurements.

- Two-sided optimal grid problem is known to be irreducible to 1D (Druskin, Moskow)
- Special choice of topology is needed
- Network with a \textit{two-sided} graph $T_n$ is proposed (left: $n = 10$)
- Network with graph $T_n$ is critical and well-connected
- Can be recovered with layer peeling
- Grids are computed using the sensitivity analysis exactly like in the pyramidal case
Sensitivity grids for the two-sided problem

Two-sided graph $T_n$ lacks the top-down symmetry. Resolution can be doubled by also fitting the data with a network turned upside-down.

Left: single optimal grid; right: double resolution grid; $n = 16$. 
Numerical results: test conductivities

Left: smooth; right: piecewise constant chest phantom.
Numerical results: smooth $\sigma +$ conformal

Left: piecewise linear; right: one step Gauss-Newton, $\beta = 0.65\pi$, $n = 17$, $\omega_0 = -\pi/10$. 
Numerical results: smooth $\sigma +$ quasiconformal

Left: piecewise linear; right: one step Gauss-Newton,

$\beta = 0.65\pi$, $K = 0.65$, $n = 17$, $\omega_0 = -\pi/10$. 
Numerical results: smooth $\sigma +$ pyramidal

Left: piecewise linear; right: one step Gauss-Newton,

$\beta = 0.65\pi$, $n = 16$, $\omega_0 = -\pi/10$. 
Numerical results: smooth $\sigma +$ two-sided

Left: piecewise linear; right: one step Gauss-Newton, $n = 16$, $B_A$ is solid red.
Numerical results: piecewise constant $\sigma +$ conformal

Left: piecewise linear; right: one step Gauss-Newton,

$\beta = 0.65\pi$, $n = 17$, $\omega_0 = -3\pi/10$. 
Numerical results: piecewise constant $\sigma +$ quasiconf.

Left: piecewise linear; right: one step Gauss-Newton,

$\beta = 0.65\pi, K = 0.65, n = 17, \omega_0 = -3\pi/10.$
Numerical results: piecewise constant $\sigma +$ pyramidal

Left: piecewise linear; right: one step Gauss-Newton,

$\beta = 0.65\pi$, $n = 16$, $\omega_0 = -3\pi/10$. 
Numerical results: piecewise constant $\sigma$ + two-sided

Left: piecewise linear; right: one step Gauss-Newton, $n = 16$, $B_A$ is solid red.
Numerical results: high contrast conductivity

- We solve the full non-linear problem
- No artificial regularization
- No linearization
- Big advantage: can capture really high contrast behavior
- Test case: piecewise constant conductivity, contrast $10^4$
- Most existing methods fail
- Our method: relative error less than 5% away from the interface

Test conductivity, contrast $10^4$. 
High contrast reconstruction, \( n = 14, \ \omega_0 = -11\pi/20, \) contrast \( 10^4 \).
Left: reconstruction; right: pointwise relative error.
Numerical results: EIT in the half plane

Can be used in different domains. Example: half plane, smooth $\sigma$.

Left: true; right: reconstruction, $n = 16$. 

$4.0\%$
Numerical results: EIT in the half plane

Can be used in different domains. Example: half plane, layered $\sigma$.

Left: true; right: reconstruction, $n = 16$. 

21.1%
Conclusions

Two distinct computational approaches to the partial data EIT:

1. Circular networks and (quasi)conformal mappings
   - Uses existing theory of optimal grids in the unit disk
   - Tradeoff between the uniform resolution and anisotropy
   - Conformal: isotropic solution, rigid electrode positioning, grid clustering leads to poor resolution
   - Quasiconformal: artificial anisotropy, flexible electrode positioning, uniform resolution, some distortions
   - Geometrical distortions can be corrected by preconditioned Gauss-Newton

2. Sensitivity grids and special network topologies (pyramidal, two-sided)
   - No anisotropy or distortions due to (quasi)conformal mappings
   - Theory of discrete inverse problems developed
   - Sensitivity grids work well
   - Independent of the domain geometry
References


