Data-to-Born transform for multiple removal, inversion and imaging with waves

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Introduction

- **Inversion with waves**: determine properties of a medium in the bulk from response measured at or near the surface.

- **Highly nonlinear** problem due to, in part, *multiple scattering*.

- Given the full waveform response, can we compute the response of the *same medium* if waves propagated in the *single scattering* regime, i.e. in *Born regime*?

- Turns out we can!

- A highly nonlinear transform takes full waveform data to single scattering data: **Data-to-Born (DtB) transform**.

- Can use as preprocessing step and integrate into **existing workflows**.
Forward model

- Generic wave equation: DtB works for both **acoustics** and **elasticity** (also **electromagnetics**):

\[ \partial_t^2 P(t, x) + L_q L_q^T P(t, x) = 0, \quad x \in \Omega, \quad t > 0, \]

here \( L_q \) is a **first order** differential operator, \( q \) is the **reflectivity**

- Model \( m \) **shots** with corresponding wavefields in a single matrix

\[ P(t, x) = \begin{bmatrix} P^{(1)}(t, x), \ldots, P^{(m)}(t, x) \end{bmatrix} \]

- Shots modeled by **initial conditions**

\[ P(0, x) = b(x) = \begin{bmatrix} b^{(1)}(x), \ldots, b^{(m)}(x) \end{bmatrix}, \quad \partial_t P(0, x) = 0 \]

- Solution

\[ P(t, x) = \cos \left( t \sqrt{L_q L_q^T} \right) b(x) \]
Collocated sources and receivers: receiver matrix is also $b(x)$

Data is **sampled** in time at $2n$ instants $t_k = k\tau$, close to **Nyquist** rate

**Data model** becomes

$$D_k = \int_{\Omega} b(x)^T \cos \left( t \sqrt{L_q^T L_q} \right) b(x) \, dx \in \mathbb{R}^{m \times m}, \quad k = 0, 1, \ldots, 2n-1,$$

or simply

$$D_k = \int_{\Omega} b(x)^T P_k(x) \, dx \in \mathbb{R}^{m \times m},$$

where

$$P_k(x) = P(t_k, x) = \cos \left( k\tau \sqrt{L_q^T L_q} \right) b(x)$$

are **wavefield snapshots**
The propagator

- Important object: **propagator** operator

\[ P_q = \cos \left( \tau \sqrt{L_q L_q^T} \right), \]

think of it as **Green’s function**

- Using propagator, snapshots admit representation

\[ P_k = T_k(P_q)b, \quad k = 0, 1, \ldots, 2n - 1, \]

via **Chebyshev polynomials** \( T_k \)

- **Notation**: let \( T \) denote both transpose and \( L_2(\Omega) \) inner product, then the **data model** becomes

\[ D_k = b^T P_k = b^T T_k(P_q)b, \quad k = 0, 1, \ldots, 2n - 1 \]
Reduced order model (ROM)

- Obviously, impossible to find $P_q$ from finite data $D_k \in \mathbb{R}^{m \times m}$, $k = 0, 1, \ldots, 2n - 1$
- What can we find? Reduced order model (ROM) for $P_q$!
- Specifically, projection ROM

$$\widetilde{P}_q = V^T P_q V \in \mathbb{R}^{nm \times nm}, \quad \widetilde{b} = V^T b \in \mathbb{R}^{nm \times m},$$

where “columns” of $V$ form orthonormal basis for some subspace
- Of course, ROM must fit the data

$$D_k = b^T \mathcal{T}_k(P_q)b = \widetilde{b}^T \mathcal{T}_k(\widetilde{P}_q)\widetilde{b}, \quad k = 0, 1, \ldots, 2n - 1$$
- Data interpolation uniquely defines projection (Krylov) subspace

range($\Pi$),

spanned by snapshots, “columns” of snapshot matrix

$$\Pi = [P_0, P_1, \ldots, P_{n-1}]$$
Mass and stiffness matrices from data

- If we knew **internal data**, snapshots $\Pi$, we could **orthogonalize** them to find
  \[ V = \begin{bmatrix} V_0, V_1, \ldots, V_{n-1} \end{bmatrix} \]

- **Multiplicative** property of Chebyshev polynomials to the rescue!
  \[ T_j(x)T_k(x) = \frac{1}{2} \left[ T_{j+k}(x) + T_{|j-k|}(x) \right] \]

- Recall snapshots and data
  \[ P_k = T_k(\mathcal{P}_q)b, \quad D_k = b^T T_k(\mathcal{P}_q)b \]

- Can find **inner products** from the data:
  \[ (\Pi^T \Pi)_{j,k} = P_j^T P_k = \frac{1}{2} \left[ D_{j+k} + D_{|j-k|} \right] \]
  \[ (\Pi^T \mathcal{P}_q \Pi)_{j,k} = P_j^T \mathcal{P}_q P_k = \frac{1}{4} \left[ D_{j+k+1} + D_{|j+k-1|} + D_{|j-k+1|} + D_{|j-k-1|} \right] \]
Orthogonalized snapshots $V$ can be related to $\Pi$ via block Gram-Schmidt orthogonalization (block QR factorization)

$$\Pi = VR, \quad V = \Pi R^{-1},$$

with block upper triangular $R$ ($m \times m$ blocks)

Then

$$\Pi^T \Pi = R^T R$$

is block Cholesky factorization of mass matrix $\Pi^T \Pi$ known from the data

Finally, projection ROM is given by

$$\tilde{\mathcal{P}}_q = V^T \mathcal{P}_q V = R^{-T} (\Pi^T \mathcal{P}_q \Pi) R^{-1},$$

with both $R$ and stiffness matrix $\Pi^T \mathcal{P}_q \Pi$ known from data
ROM properties

- ROM computation is entirely **data-driven**, no a priori information on continuum problem needed
- Gram-Schmidt orthogonalization (Cholesky) preserves **causality**: only looks backwards in time
- Reduced order propagator $\widetilde{P}_q$ is **block tridiagonal**, blocks correspond to **layers of equal travel time** from the source array, can be seen as a (block) **second-order difference scheme**
- Orthogonalized snapshots $V$ depend on the medium only **kinematically**, **reflections** are effectively **suppressed** in $V$ (will see later in numerics)
- A version **robust** to noise and modeling errors exists: based on **spectral truncation** of the mass matrix $\Pi^T\Pi$, block Cholesky replaced with **block Lanczos**
Second order difference formulation

- We computed ROM propagator $\tilde{P}_q$, can we find reduced model for $L_q$ itself?
- Wavefield snapshots satisfy exactly the second order difference scheme

$$\frac{P_{k+1} - 2P_k + P_{k-1}}{\tau^2} + L_qL_q^TP_k = 0, \quad k \geq 0,$$

$$P_0 = b, \quad P_{-1} = P_1,$$

with

$$\frac{2}{\tau^2}(I - P_q) = L_qL_q^T$$

- Can show

$$L_q = L_q + O(\tau^2)$$

- This construction has a reduced order analogue
ROM propagator factorization

- Reduced order snapshots $\tilde{\mathbf{P}}_k = \mathcal{T}_k(\tilde{\mathbf{P}}_q)\tilde{\mathbf{b}}$ also satisfy a second order scheme

$$\frac{\tilde{\mathbf{P}}_{k+1} - 2\tilde{\mathbf{P}}_k + \tilde{\mathbf{P}}_{k-1}}{\tau^2} + \tilde{\mathbf{L}}_q \tilde{\mathbf{L}}_q^T \tilde{\mathbf{P}}_k = 0, \quad k \geq 0,$$

$$\tilde{\mathbf{P}}_0 = \tilde{\mathbf{b}} = \text{RE}_1, \quad \tilde{\mathbf{P}}_{-1} = \tilde{\mathbf{P}}_1,$$

- To compute $\tilde{\mathbf{L}}_q$ perform second block Cholesky factorization

$$\frac{2}{\tau^2} (\mathbf{I} - \tilde{\mathbf{P}}_q) = \tilde{\mathbf{L}}_q \tilde{\mathbf{L}}_q^T$$

- So we have $\tilde{\mathbf{L}}_q \in \mathbb{R}^{nm \times nm}$, a finite dimensional approximation of $L_q$

- Since $\tilde{\mathbf{P}}_q$ is block tridiagonal, $\tilde{\mathbf{L}}_q$ is block lower bi-diagonal

- Why is $\tilde{\mathbf{L}}_q$ useful?
Example: acoustic wave equation

Consider acoustic wave equation for pressure $p(t, x)$ in the form

$$\partial_t^2 p(t, x) - \sigma(x)c(x)\nabla \cdot \left[ \frac{c(x)}{\sigma(x)} \nabla p(t, x) \right] = 0,$$

with velocity $c(x)$ and impedance $\sigma(x)$.

Assume kinematics is known, seek Born approximation with respect to perturbation of $\sigma(x)$.

Liouville transform converts wave equation to first order system

$$\partial_t \begin{pmatrix} P(t, x) \\ \hat{P}(t, x) \end{pmatrix} = \begin{pmatrix} 0 & -L_q \\ L_q^T & 0 \end{pmatrix} \begin{pmatrix} P(t, x) \\ \hat{P}(t, x) \end{pmatrix},$$

with corresponding second order form

$$\partial_t^2 P(t, x) + L_qL_q^T P(t, x) = 0$$
The reflectivity

- The operators $L_q$ and $L_q^T$ are given by

$$L_q = -\sqrt{c(x)} \nabla \cdot \sqrt{c(x)} + \frac{c(x)}{2} [\nabla q(x)],$$

$$L_q^T = \sqrt{c(x)} \nabla \sqrt{c(x)} + \frac{c(x)}{2} [\nabla q(x)],$$

with reflectivity $q(x) = \ln \sigma(x)$

- If $c(x)$ is known and fixed, then $L_q$ and $L_q^T$ are affine in $q(x)$

- Since

$$\tilde{L}_q \approx L_q,$$

then $\tilde{L}_q$ is approximately affine in reflectivity $q(x)$!

- **Perturbing** with respect to $q(x)$ becomes easy!
First order reduced order system

- **Reduced order** analogue of the **first order system**

\[
\begin{align*}
\tilde{P}_{k+1} - \tilde{P}_k &= -\tilde{L}_q \tilde{P}_k, \quad k = 0, \ldots, 2n - 2, \\
\tilde{P}_k - \tilde{P}_{k-1} &= \tilde{L}_q^T \tilde{P}_k, \quad k = 1, \ldots, 2n - 1,
\end{align*}
\]

with initial conditions

\[
\begin{align*}
\tilde{P}_0 &= \tilde{b}, & \tilde{P}_0 + \tilde{P}_{-1} &= 0.
\end{align*}
\]

- The right hand side is **approximately affine** in \( q(x) \)

- Perturbing \( \tilde{L}_q \) with respect to \( q \) simply gives

\[
\delta \tilde{L} = \tilde{L}_q - \tilde{L}_0,
\]

where \( \tilde{L}_0 \) is computed in **reference medium** with \( q \equiv 0 \)
Data-to-Born transform

- Born approximation is a linearized perturbation
- **Perturbed** reduced order first order system

\[
\begin{align*}
\delta \tilde{P}_{k+1} - \delta \tilde{P}_k &= -\tilde{L}_0 \delta \tilde{P}_k - (\tilde{L}_q - \tilde{L}_0) \tilde{P}_{0,k}, \quad k = 0, \ldots, 2n - 2, \\
\delta \tilde{P}_k - \delta \tilde{P}_{k-1} &= \tilde{L}_0^T \delta \tilde{P}_k + (\tilde{L}_q^T - \tilde{L}_0^T) \tilde{P}_{0,k}, \quad k = 1, \ldots, 2n - 1,
\end{align*}
\]

with initial conditions

\[
\delta \tilde{P}_0 = 0, \quad \delta \tilde{P}_0 + \delta \tilde{P}_{-1} = 0
\]

- Here $\tilde{P}_{0,k}, \tilde{P}_{0,k}$ are reduced order snapshots in reference media
- Data-to-Born transform is

\[
D_{k}^{DtB} = D_{0,k} + \tilde{b}^T \delta \tilde{P}_k, \quad k = 0, 1, \ldots, 2n - 1,
\]

compare to full waveform data $D_k = \tilde{b}^T \tilde{P}_k$
Numerical results: Acoustic snapshots

- Array with $m = 50$ sensors
- Snapshots plotted for a single source
Numerical results: Acoustic true Born vs. DtB

- Single row of data matrix corresponding to source
- **Vertical:** time (in units of $\tau$)
- **Horizontal:** receiver index (out of $m = 50$)

**Full waveform data**

**True Born data**

**DtB**
Numerical results: Acoustic DtB + RTM

- **Reverse time migration (RTM)**
  - Image computed from both measured full waveform data and DtB transformed data
Numerical results: Elasticity, two cracks

- Transform elasticity problem to first order form: **Liouville transform**
- If both velocities are fixed (here $c_p = 2c_s$), there is only one **independent impedance** $\sigma_p$
- **Source**: horizontal force, $m = 25$

**Full waveform data**

**True Born data**

**DtB**

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Data-to-Born transform
Numerical results: Elasticity, salt dome

- Transform elasticity problem to first order form: **Liouville transform**
- If both velocities are fixed (here $c_p = 2c_s$), there is only one independent impedance $\sigma_p$
- Source: horizontal force, $m = 25$
Conclusions and future work

- **Data-to-Born**: transform full waveform data to single scattered Born data for the same medium
- Based on techniques of **model order reduction**
- **Data-driven** approach relying on classical **linear algebra** algorithms (Cholesky, Lanczos), no computations in the continuum
- Works for all linear waves: **acoustic, elastic, electromagnetic**
- Easy to integrate into **existing workflows** as a preprocessing step
- Enables the use of **linearized inversion** algorithms

**Future work:**
- Test linearized inversion (e.g. **LS-RTM**) on DtB data
- Extend to **frequency domain** wave equation (Helmholtz)
- Use DtB-like approach to extract **higher orders of scattering** from full waveform data


Related work:
