8.6 Test of the equality of several means

1. ANOVA for one factor:

ANOVA: The name of ANOVA stands for analysis of variance.

Goal: Consider \( m \) normal distributions with unknown means \( \mu_1, \ldots, \mu_m \) and an unknown but common variance \( \sigma^2 \). We wish to test the equality of the \( m \) means. This method is called ANOVA (analysis of variance) for one factor. This method derives its name from the fact that \( SS(TO) = (n - 1)S^2 \) (see below for explanation).

Consider \( X_1 \sim N(\mu_1, \sigma^2), \ldots, X_m \sim N(\mu_m, \sigma^2) \).

Null Hypothesis: \( H_0 : \mu_1 = \mu_2 = \ldots = \mu_m = \mu \)

Alternative Hypothesis: All possible alternative hypothesis.

Let \( X_{i1}, \ldots, X_{in_i} \) be independent random samples from \( N(\mu_i, \sigma^2), i = 1, 2, \ldots, m \). Let \( n = n_1 + \ldots + n_m \).

- Means: \( \bar{X}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij}, i = 1, \ldots, m. \)

- Grand mean: \( \bar{X}_. = \frac{1}{n} \sum_{i=1}^{m} \sum_{j=1}^{n_i} X_{ij} \)

See the following table:

<table>
<thead>
<tr>
<th></th>
<th>Means</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( X_{1} )</td>
<td>( X_{11} )</td>
<td>( X_{12} )</td>
<td>\ldots</td>
<td>( X_{1n_1} )</td>
</tr>
<tr>
<td></td>
<td>( X_{2} )</td>
<td>( X_{21} )</td>
<td>( X_{22} )</td>
<td>\ldots</td>
<td>( X_{2n_2} )</td>
</tr>
<tr>
<td></td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
</tr>
<tr>
<td></td>
<td>( X_{m} )</td>
<td>( X_{m1} )</td>
<td>( X_{m2} )</td>
<td>\ldots</td>
<td>( X_{mn_m} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Grand Mean</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \bar{X}_. )</td>
<td></td>
</tr>
</tbody>
</table>

- Variations:

\[
SS(E) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2 \]
n measures the variation within each group
\[ SS(T) = \sum_{i=1}^{m} n_i (\bar{X}_i - \bar{X}.)^2 \] measures the variation among the treatment means.

\[ SS(TO) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}.)^2 \] measures the total variation.

Note that \( SS(TO) = (n-1)S^2 \), where \( S^2 \) is the sample variance. The name “ANOVA” comes from this fact. We have

\[ SS(TO) = SS(T) + SS(E). \]

**Test Statistics:**

\[ F = \frac{SS(T)/(m-1)}{SS(E)/(n-m)} \sim F(m-1, n-m) \]

**Critical Region:** We would reject \( H_0 \) if \( F_{obs} \) is too large, i.e., the critical region is \( F \geq c \), where, by the standard method, \( c = F_{\alpha}(m-1, n-m) \).

**Computational Formula:**

\[
SS(TO) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} X_{ij}^2 - \frac{1}{n} \left( \sum_{i=1}^{m} \sum_{j=1}^{n_i} X_{ij} \right)^2
\]

\[
SS(T) = \sum_{i=1}^{m} \frac{1}{n_i} \left( \sum_{j=1}^{n_i} X_{ij} \right)^2 - \frac{1}{n} \left( \sum_{i=1}^{m} \sum_{j=1}^{n_i} X_{ij} \right)^2
\]

\[
SS(E) = SS(TO) - SS(T).
\]

**Why \( F = \frac{SS(T)/(m-1)}{SS(E)/(n-m)} \sim F(m-1, n-m) \)?**

Clearly, \( SS(TO)/\sigma^2 \) is \( \chi^2(n-1) \), and \( SS(E)/\sigma^2 \) is \( \chi^2(\sum_{i=1}^{m} (n_i - 1)) = \chi^2(n-m) \). Since \( SS(TO) = SS(T) + SS(E) \), we have \( SS(T) \sim \chi^2(m-1) \). Thus

\[
F = \frac{SS(T)/(m-1)}{SS(E)/(n-m)} = \frac{[SS(T)/\sigma^2]/(m-1)}{[SS(E)/\sigma^2]/(n-m)} \sim F(m-1, n-m).
\]

**One Factor ANOVA table.**

<table>
<thead>
<tr>
<th>Source</th>
<th>Sum of Squares</th>
<th>D.F.</th>
<th>Mean Square</th>
<th>F-ration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatment:</td>
<td>SS(T)</td>
<td>( m-1 )</td>
<td>( MS(T) = \frac{SS(T)}{m-1} )</td>
<td>( MS(T) / MS(E) )</td>
</tr>
<tr>
<td>Error:</td>
<td>SS(E)</td>
<td>( n-m )</td>
<td>( MS(E) = \frac{SS(E)}{n-m} )</td>
<td></td>
</tr>
<tr>
<td>Total:</td>
<td>SS(TO)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2
2. ANOVA for two factors:

**Goal:** Consider a situation in which it is desirable to investigate the effects of two factors that influence the outcome of an experiment. For example, a teaching method and size of a class might influence a student’s score on a standard test.

(1) ANOVA for two factors, one observation per cell.

Assume that there are two factors (attributes), one of which has \(a\) levels and the other \(b\) level. There are thus \(n = ab\) possible combinations, each of which determines a cell. Let us think of these cells as being arranged in \(a\) rows and \(b\) columns. Here we take one observation per cell, and we denote the observation in the \(i\)th row and \(j\)th column by \(X_{ij}\). We have the following table:

<table>
<thead>
<tr>
<th>Factor A</th>
<th>Level 1</th>
<th>Level 2</th>
<th>...</th>
<th>Levelb (\bar{X}_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Level 1</td>
<td>(X_{11})</td>
<td>(X_{12})</td>
<td>...</td>
<td>(X_{1b})</td>
</tr>
<tr>
<td>Level 2</td>
<td>(X_{21})</td>
<td>(X_{22})</td>
<td>...</td>
<td>(X_{2b})</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>Level (a)</td>
<td>(X_{a1})</td>
<td>(X_{a2})</td>
<td>...</td>
<td>(X_{ab})</td>
</tr>
<tr>
<td>(\bar{X}_j)</td>
<td>(\bar{X}_1)</td>
<td>(\bar{X}_2)</td>
<td>...</td>
<td>(\bar{X}_b)</td>
</tr>
</tbody>
</table>

Let \(X_{ij} \sim N(\mu_{ij}, \sigma^2), i = 1, \ldots, a\) and \(j = 1, \ldots, b\) be the random variables denoting the measurement when factor \(A\) is held at level \(i\) and factor \(B\) is held at level \(j\). We shall assume that the means are composed of a row effect, a column effect, and an overall effect in some additive way, namely \(\mu_{ij} = \mu + \alpha_i + \beta_j\) where \(\sum_{i=1}^{a} \alpha_i = 0\) and \(\sum_{j=1}^{b} \beta_j = 0\). The parameter \(\alpha_i\) represents the \(i\)th row effect, and the parameter \(\beta_j\) represents the \(j\)th row effect.

**Null Hypotheses:** There is no row effect: \(H_A : \alpha_1 = \alpha_2 = \cdots = \alpha_a = 0\) against all possible alternative hypotheses

**Or Null Hypotheses:** There is no column effect: \(H_B : \beta_1 = \beta_2 = \cdots = \beta_b = 0\) against all possible alternative hypotheses.

Let

\[
\bar{X}_i = \frac{1}{b} \sum_{j=1}^{b} X_{ij}, \text{ the average of measurements obtained when } A \text{ is held at the level } i
\]

\[
\bar{X}_j = \frac{1}{a} \sum_{i=1}^{a} X_{ij}, \text{ the average of measurements obtained when } B \text{ is held at the level } j
\]
\[ X_\cdot = \frac{1}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} X_{ij}, \quad \text{the grand mean} \]

\[ SS(A) = b \sum_{i=1}^{a} (\bar{X}_i - \bar{X}_\cdot)^2 \quad (a - 1 \text{ degrees of freedom}) \]

\[ SS(B) = a \sum_{j=1}^{b} (\bar{X}_j - \bar{X}_\cdot)^2 \quad (b - 1 \text{ degrees of freedom}) \]

\[ SS(E) = \sum_{i=1}^{a} \sum_{j=1}^{b} (X_{ij} - \bar{X}_i - \bar{X}_j + \bar{X}_\cdot)^2 \quad ((a - 1)(b - 1) \text{ degrees of freedom}) \]

\[ SS(TO) = \sum_{i=1}^{a} \sum_{j=1}^{b} (X_{ij} - \bar{X}_\cdot)^2 = SS(A) + SS(B) + SS(E). \quad (ab - 1 \text{ degrees of freedom}) \]

**Test Statistics:** For \( H_A: \)

\[ F = \frac{SS(A)/(a - 1)}{SS(E)/[(a - 1)(b - 1)]} \sim F(a - 1, (a - 1)(b - 1)) \]

For \( H_B: \)

\[ F = \frac{SS(B)/(b - 1)}{SS(E)/[(a - 1)(b - 1)]} \sim F(b - 1, (a - 1)(b - 1)) \]

**Critical Region:** We would reject \( H_A \) (respectively \( H_B \)) if \( F_{obs} \) is too large, i.e., the critical region is \( F \geq c \), where, by the standard method, \( c = F_{\alpha}(a - 1, (a - 1)(b - 1)) \) (resp. \( c = F_{\alpha}(b - 1, (a - 1)(b - 1)) \)).

**Two Factor ANOVA table, one observation per cell.**

<table>
<thead>
<tr>
<th>Source</th>
<th>Sum of Squares</th>
<th>D.F.</th>
<th>Mean Square</th>
<th>F-ration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factor A(row):</td>
<td>SS(A)</td>
<td>( a - 1 )</td>
<td>( MS(A) = \frac{SS(A)}{a - 1} )</td>
<td>( MS(A) )</td>
</tr>
<tr>
<td>Factor B(Column):</td>
<td>SS(B)</td>
<td>( b - 1 )</td>
<td>( MS(A) = \frac{SS(B)}{b - 1} )</td>
<td>( MS(B) )</td>
</tr>
<tr>
<td>Error:</td>
<td>SS(E)</td>
<td>( (a - 1)(b - 1) )</td>
<td>( MS(E) = \frac{SS(E)}{(a - 1)(b - 1)} )</td>
<td></td>
</tr>
<tr>
<td>Total:</td>
<td>SS(TO)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(2) ANOVA for two factors, \( c \) observation per cell.

In this case, we shall consider a two-way classification problem with interaction, which \( c > 1 \) independent observations are taken.
Null Hypotheses: No row, column, and interaction effects.

Test Statistics: For \( H_A: \alpha_1 = \alpha_2 = \cdots = \alpha_a = 0 \) (no row effect):

\[
F = \frac{SS(A)/(a-1)}{SS(E)/(ab(c-1))} \sim F(a-1, ab(c-1))
\]

For \( H_B: \beta_1 = \cdots = \beta_b = 0 \) (no column effect)

\[
F = \frac{SS(B)/(b-1)}{SS(E)/(ab(c-1))} \sim F(b-1, ab(c-1))
\]

For \( H_C: \gamma_{ij} = 0, i = 1, 2, \ldots a, j = 1, \ldots, b \) (no interaction effect)

\[
F = \frac{SS(AB)/[(a-1)(b-1)]}{SS(E)/(ab(c-1))} \sim F((a-1)(b-1), ab(c-1))
\]

Critical Region: We would reject \( H_A \) (respectively \( H_B, H_C \)) if \( F_{obs} \) is too large, i.e., the critical region is \( F \geq c \), where, by the standard method, \( c = F_{a-1, ab(c-1)}(\alpha) \) (resp. \( c = F_{a}(b-1, ab(c-1)) \), \( c = F_{((a-1)(b-1), ab(c-1))}(\alpha) \)).

- \( SS(\text{TO}) = SS(E) + SS(A) + SS(B) + SS(AB) \).

- \( SS(\text{TO}) = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} (X_{ihk} - \bar{X}_{..})^2 \) \( (abc - 1 \text{ degrees of freedom}) \)

- \( SS(A) = bc \sum_{i=1}^{a} (\bar{X}_{i..} - \bar{X}_{..})^2 \) \( (a - 1 \text{ degrees of freedom}) \)

- \( SS(B) = ac \sum_{j=1}^{b} (\bar{X}_{.j.} - \bar{X}_{..})^2 \) \( (b - 1 \text{ degrees of freedom}) \)

- \( SS(E) = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} (X_{ijk} - \bar{X}_{ij.})^2 \) \( (ab(c-1) \text{ degrees of freedom}) \)

- \( SS(AB) = c \sum_{i=1}^{a} \sum_{j=1}^{b} (\bar{X}_{ij.} - \bar{X}_{i..} - \bar{X}_{.j.} + \bar{X}_{..})^2 \) \( ((a-1)(b-1) \text{ degrees of freedom}) \)
Two Factor ANOVA table, \( c \) observation per cell.

<table>
<thead>
<tr>
<th>Source</th>
<th>Sum of Squares</th>
<th>D.F.</th>
<th>Mean Square</th>
<th>( F )-ration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Factor A (row):</td>
<td>( SS(A) )</td>
<td>( a - 1 )</td>
<td>( MS(A) = \frac{SS(A)}{a-1} )</td>
<td>( MS(A) )</td>
</tr>
<tr>
<td>Factor B (Column):</td>
<td>( SS(B) )</td>
<td>( b - 1 )</td>
<td>( MS(A) = \frac{SS(B)}{b-1} )</td>
<td>( MS(B) )</td>
</tr>
<tr>
<td>Factor AB (Interaction):</td>
<td>( SS(AB) )</td>
<td>( (a-1)(b-1) )</td>
<td>( MS(AB) = \frac{(a-1)(b-1)}{SS(AB)} )</td>
<td>( MS(AB) )</td>
</tr>
<tr>
<td>Error:</td>
<td>( SS(E) )</td>
<td>( ab(c-1) )</td>
<td>( MS(E) = \frac{SS(E)}{ab(c-1)} )</td>
<td>( MS(E) )</td>
</tr>
<tr>
<td>Total:</td>
<td>( SS(TO) )</td>
<td>( abc-1 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>