HW1 solution

1. \[ A \cup B = \{2, 3, 4, 5, 6, 8\}, \quad A \cap B = \{4, 6\} \]

2. a, d, f, h

3. Proof: for \( x \in [-1, 1] \), i.e. \(-1 \leq x \leq 1\), we have \(-1 \leq x^3 \leq 1\). Hence \( x^3 \in [-1, 1] \). Thus \( A \subseteq B \). On the other hand, for each \( a \in B \), Take \( x = a^{1/3} \), notice that \(-1 \leq a^{1/3} \leq 1\), hence \( a = x^3 \in A \). so \( B \subseteq A \). Therefore, \( A = B \).

Remark: To prove the statement \( A = B \), try to prove \( A \subseteq B \) and \( B \subseteq A \).

4. (a) domain is \{1, 2, 3, 4\}, codomain is \( \mathbb{N} \), range is \{1, 4, 9, 16\}.

(b) yes. It is one-to-one. Here is the proof: Assume that \( f(n_1) = f(n_2) \) where \( n_1, n_2 \in \{1, 2, 3, 4\} \), i.e. \( n_1^2 = n_2^2 \). Since \( n_1, n_2 \) are both positive, this implies that \( n_1 = n_2 \). So \( f \) is one-to-one.

(c) no, because the range of \( f \) is not equal to the codomain of \( f \).

5. (a) one to one: suppose \( f(x) = f(y) \), i.e., \( x^3 = y^3 \), then \( x = y \), so \( f \) is one-to-one. onto: codomain of \( f \) is \([-1, 1]\), and the range of \( f \) is also \([-1, 1]\), so \( f \) is onto.

(b) To find \( f^{-1} : [-1, 1] \rightarrow [-1, 1] \), by the definition, For any \( y \in [-1, 1] \), take \( x \in [-1, 1] \) with \( f(x) = y \). Then \( f^{-1}(y) = x \). In our case, \( f(x) = x^3 \), so \( x^3 = y \), or \( x = y^{1/3} \). So \( f^{-1}(y) = y^{1/3} \). Usually, we use \( x \) as independent variable, and \( y \) as dependent variable, so we write \( f^{-1} \) as \( f^{-1}(x) = x^{1/3} \).

6. (1) Define \( 0 = a, 1 = b \), then F1-F5 can be verified.

(2) \( 0 = a, 1 = b, -1 = b \).

7. \( q(x) = x^3 + 5x + 6, \quad r(x) = x - 1 \).

8. No, because if \( F \) is a field, then every non-zero element must have a multiplicative inverse, but by taking \( P(x) = x \in F \), since \( \frac{1}{x} \) is not a polynomial anymore, so \( P(x) \) does not have a multiplicative inverse. Thus \( F \) is not a field.