HW3 solution

1. (c). NO, do not exist $a, b$ satisfy $a(1, 0, 1, -1) + b(0, 1, 1, 1) = (-1, 1, 1, 2)$ (g) YES. Write

$$\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + c \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$ We find $a, b, c$, we solve (by comparing the entries)

$$1 = a + c, 2 = b + c, -3 = -a, 4 = b.$$ Hence $a = 3, b = 4, c = -2$

2. To show that $\{(1,1,0),(1,0,1),(0,1,1)\}$ generates $\mathbb{F}^3$, we need to show that for every $(a, b, c) \in \mathbb{F}^3$, we can find $s_1, s_2, s_3 \in \mathbb{F}$ such that

$$(a, b, c) = s_1(1,1,0) + s_2(1,0,1) + s_3(0,1,1).$$ That is we can solve the system of linear equations (with unknowns $s_1, s_2, s_3$)

$$s_1 + s_2 = a, s_1 + s_3 = b, s_2 + s_3 = c.$$ Solving above, we get

$$s_1 = \frac{1}{2}(a + b - c), s_2 = \frac{1}{2}(a - b + c), s_3 = \frac{1}{2}(b - a + c).$$ Hence $\{(1,1,0),(1,0,1),(0,1,1)\}$ generates $\mathbb{F}^3$.

3. For $\forall P = a_n x^n + \cdots + a_0$, take $s_0 = a_0, s_1 = a_1, \ldots, s_n = x^n$, then

$$P = s_1 \cdot 1 + \cdots + s_n x^n.$$ This proves that $\{1, x, \ldots, x^n\}$ generates $P_n(F)$.

4. Proof: any vector $x \in \text{span}(S_1 \cap S_2)$ can be written as $x = a_1 v_1 + \cdots + a_n v_n$, where $v_1, \ldots, v_n \in S_1 \cap S_2$, therefore $v_1, \ldots, v_n \in S_1$ thus $x = a_1 v_1 + \cdots + a_n v_n \in \text{span}(S_1)$, similarly $x = a_1 v_1 + \cdots + a_n v_n \in \text{span}(S_2)$. So $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$.

However, $\text{span}(S_1 \cap S_2)$ and $\text{span}(S_1) \cap \text{span}(S_2)$ are not always equal. Here are examples that they are equal and not equal: Example 1: $S_1 = \{a\}, S_2 = \{a, b\}$, where $a, b$ are linearly independent, then $\text{span}(S_1 \cap S_2) = \text{span}(S_1) \cap \text{span}(S_2)$; Example 2: $S_1 = \{a\}, S_2 = \{b\}$, where $a \neq b$ are linearly dependent (for example taking $b = 2a$, then $\text{span}(S_1 \cap S_2) \neq \text{span}(S_1) \cap \text{span}(S_2)$).

5. (e) Write

$$a_1((1, -1, 2) + a_2(1, -2, 1) + a_3(1, 1, 4) = (0, 0, 0).$$
By solving the system of equations
\[ a_1 + a_2 + a_3 = 0, -a_1 - 2a_2 = 0, 2a_1 + a_2 + 4a_3 = 0 \]
we get \( a_1 = -3, a_2 = 2, a_3 = 1 \). So they are linearly dependent.

(f) Write
\[ a_1((1, -1, 2) + a_2(2, 0, 1) + a_3(-1, 2, -1)) = (0, 0, 0). \]
By solving the system of equations, we get \( a_1 = a_2 = a_3 = 0 \). So it is linearly independent.

(i) Write
\[ a_1(x^4 - x^3 + 5x^2 - 8x + 6) + a_2(-x^4 + x^3 - 5x^2 + 5x - 3) + a_3(x^4 + 3x^2 - 3x + 5) \]
\[ + a_4(2x^4 + 3x^3 + 4x^2 - x + 1) + a_5(x^3 - x + 2) = 0. \]
So we get system of equations
\[ a_1 - a_2 + a_3 + 2a_4 = 0, ..., 6a_1 - 3a_2 + 5a_3 + a_4 + 2a_5 = 0. \]
By solving this, we get \( a_1 = \cdots = a_5 = 0 \). Hence it is linearly independent.

6. (a) This is a "two-way" statement. We first prove \( \Rightarrow \), i.e. assume that \( \{u, v\} \) is linearly independent, we prove that \( \{u + v, u - v\} \) is also linearly independent. To do so, Write \( a_1(u + v) + a_2(u - v) = 0 \). We need to show that \( a_1 = a_2 = 0 \). In fact, from \( a_1(u + v) + a_2(u - v) = 0 \), we get \( (a_1 + a_2)u + (a_1 - a_2)v = 0 \). Since \( \{u, v\} \) is linearly independent, we get \( a_1 + a_2 = 0, a_1 - a_2 = 0 \) which implies that \( a_1 = a_2 = 0 \). This proves that \( \{u + v, u - v\} \) is linearly independent.

We now prove \( \Leftarrow \). Let \( a_1u + a_2v = 0 \). We need to show that \( a_1 = a_2 = 0 \). Notice that
\[ u = \frac{1}{2}((u + v) + (u - v)), \quad v = \frac{1}{2}((u + v) - (u - v)). \]
Then \( a_1u + a_2v = 0 \) becomes
\[ \frac{a_1}{2}((u + v) + (u - v)) + \frac{a_2}{2}((u + v) - (u - v)) = 0. \]
i.e.
\[ \frac{a_1 + a_2}{2}(u + v) + \frac{a_1 - a_2}{2}(u - v) = 0. \]
Since \( \{u + v, u - v\} \) is linearly independent, we get \( \frac{a_1 + a_2}{2} = 0, \frac{a_1 - a_2}{2} = 0 \) which implies that \( a_1 = a_2 = 0 \). This proves that \( \{u, v\} \) is linearly independent.

(b) can be proves in a similar way.
7. (b) Not a basis because they are linearly dependent.  
(d) It is a basis: We first check it is linearly independent (omitted), and then we check \( \text{span}\{(-1, 3, 1), (2, -4, -3), (-2, -10, -2)\} = \mathbb{R}^3 \) (omitted).

8. \((a_1, a_2, a_3, a_4) = a_1u_1 + (a_2 - a_1)u_2 + (a_3 - a_2)u_3 + (a_4 - a_3)u_4.\)

9. To show that \(\{u + v, au\}\) is a basis, we need to show that (1) It is linearly independent, (2) it spans \(V\). We first show that \(\{u + v, au\}\) is linearly independent (the method is the same as we did above in #6). Assume that \(a_1(u + v) + a_2(au) = 0, i.e. (a_1 + aa_2)u + a_1v = 0.\) Since \(\{u, v\}\) is linearly independent, we have \(a_1 + aa_2 = 0, a_1 = 0.\) Using \(a \neq 0,\) we get \(a_1 = a_2 = 0.\) So \(\{u + v, au\}\) is linearly independent.

We now show it spans. Since \(\{u, v\}\) is a basis, \(\text{span}\{u, v\} = V.\) On the other hand, since \(u = \frac{1}{a}(au), v = (u + v) - \frac{1}{a}(au),\) we have \(V = \text{span}\{u, v\} \subseteq \text{span}\{u + v, au\} \subseteq V.\) Therefore, \(\{u + v, au\}\) is a basis. Similarly, we can prove that \(\{au, bv\}\) is also a basis.

10. By solving the given system, we get \(x_2 = x_3, x_1 = x_3.\) So the subspace (the solution space) is \(W = \{(x_3, x_3, x_3) = x_3(1, 1, 1) \mid x_3 \in \mathbb{R}\}.\) Hence \((1, 1, 1)\) is a basis.

11. (3) Verify the linearity by definition. Bases for \(N(T)\) and \(R(T)\) are \(\emptyset\) and \(\{(1, 0, 2), (1, 0, -1)\},\) nullity and rank of \(T\) are 0 and 2. \(T\) is one-to-one but not onto. (5) Verify the linearity by definition. Bases for \(N(T)\) and \(R(T)\) are \(\emptyset\) and \(\{x, x^2 + 1, x^3 + 2x\},\) nullity and rank of \(T\) are 0 and 3. \(T\) is one-to-one but not onto. (6) Verify the linearity by definition. Bases for \(N(T)\) and \(R(T)\) are \(\{(1, 0, 2), (1, 0, -1)\}\) and \(\{1\},\) nullity and rank of \(T\) are \(n^2 - 1\) and 1. \(T\) is not one-to-one but onto.

12. We first calculate \(T(2, 3).\) Since \((2, 3) = -(1, 0) + 3(1, 1), T(2, 3) = T(-(1, 0) + 3(1, 1)) = -T(1, 0) + 3T(1, 1) = -(1, 4) + 3(2, 5) = (5, 11).\) Similar for \(\forall v \in \mathbb{R}^2,\) write \(v = x(1, 0) + y(1, 1),\) then \(T(v) = xT(1, 0) + yT(1, 1) = x(1, 4) + y(2, 5).\)
To see whether $T$ is one-to-one. To see whether $T$ is one-to-one, we only need to find $N(T)$. For $\forall v \in N(T)$, i.e. $T(v) = (0,0)$, so $x(1,4) + y(2,5) = 0$. Now since $(1,4), (2,5)$ are linearly independent, we have $x = y = 0$. Hence $x = 0$. So $N(T) = \{ (0,0) \}$. Thus $T$ is one-to-one (by Theorem 2.4 on P. 71.)