1. Problem 1 on Page 179.

Solution: (a) False.
(b) False.
(c) True, since $(0, ..., 0)$ is always a solution for a homog system.
(d) False.
(e) False.
(f) False.
(g) True, the solution is $x = A^{-1}b$.
(h) False. Only for homog system, the solution set is a subspace.

2. Problem 2 (b), (d), (f) Page 180.

Solution: 2(b): By eliminating $x_1$, we get $x_2 = \frac{2}{3}x_3$, and $x_1 = x_3 - x_2 = \frac{1}{3}x_3$. So the solution is $((1/3)x_3, (2/3)x_3, x_3) = x_3(1/3, 2/3, 1)$ for $x_3 \in \mathbb{R}$. A basis for the solution is can be $\{(1, 2, 3)\}$. The dimension of the solution space is 1.

2(d): By adding the first and second equation, we get $x_1 = 0$, we also get $x_2 = x_3$. So the solution set is $\{(0, x_2, x_2) = x_2(0, 1, 1), x_2 \in \mathbb{R}\}$. A basis for the solution is can be $\{(0, 1, 1)\}$. The dimension of the solution space is 1.

2(f): The solution set is $\{(0, 0)\}$, there is no basis for it. The dimension of the solution space is zero.

3. Problem 3 (b), (d), (f) Page 180.

Solution: 3(b): First we can find a particular solution for the system which is $(2/3, 1/3, 0)$. Hence, using 2(b), the solution is $(x_1, x_2, x_3) = (2/3, 1/3, 0) + t(1, 2, 3)$ where $t \in \mathbb{R}$.

3(d): First we can find a particular solution for the system which is $(2, 1, 0)$. Hence, using 2(b), the solution is $(x_1, x_2, x_3) = (2, 1, 0) + t(0, 1, 1)$ where $t \in \mathbb{R}$.

3(f): First we can find a particular solution for the system which is $(2, 1)$. Hence, using 2(b), the solution is $(x_1, x_2) = (2, 1, 0)$, i.e. the system has a unique solution.

Solution: 4(b):

\[ A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & -2 & 1 \end{pmatrix}, \]

so

\[ A^{-1} = \begin{pmatrix} 1/3 & 0 & 1/3 \\ 11/9 & 1/3 & -2/9 \\ -4/9 & 2/3 & -1/9 \end{pmatrix}. \]

The solution is

\[ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1/3 & 0 & 1/3 \\ 11/9 & 1/3 & -2/9 \\ -4/9 & 2/3 & -1/9 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}. \]

5. Problem 6 Page 180.

Solution: To find \( T^{-1}(1, 11) \) (Note that \( T \) is not one-to one, here the \( T^{-1}(1, 11) \) means the set \( \{(a, b, c) \mid T(a, b, c) = (1, 11)\} \)). we need to find \( (a, b, c) \) with \( T(a, b, c) = (1, 11) \), i.e. we need to solve \( a + b = 1, 2a - c = 11 \). By solving this system, we get

\[ (a, b, c) = (11/2, -9/2, 0) + t(1, -1, 2), t \in \mathbb{R}. \]

So

\[ T^{-1}(1, 11) = \{(11/2, -9/2, 0) + t(1, -1, 2), t \in \mathbb{R}\}. \]


Solution: (a) \((1, 3, -2) \in R(T)\) means that we can solve \( a + b = 1, b - 2c = 3, a + 2c = -2 \), this system has a solution, for example \( a = -2, b = 3, c = 0 \), so it is true that \((1, 3, -2) \in R(T)\).

(b): No, \((2, 1, 1)\) is also in \( R(T) \), since the system \( a + b = 2, b - 2c = 1, a + 2c = 1 \) has a solution \( a = 1, b = 1, c = 0 \).
7. Problem 9 on Page 181.

Proof: By definition, \( b \in R(L_A) \) if and only if there exists \( x \in \mathbb{R}^n \) such that \( L_A(x) = b \). By the definition of \( L_A \), this means that \( Ax = b \). So \( Ax = b \) has a solution if and only if \( b \in R(L_A) \).


Proof: The statement is true. Here is the proof: From Theorem 3.11, \( Ax = b \) has a solution if and only \( rank(A) = rank(A|b) \), So we only need to show that \( rank(A|b) = m \). In fact, since the span of the columns of the matrix \( A \) is a subset of the space of the columns of \( (A|b) \), and the rank of a matrix is the dimension of the space spanned by its column vectors, we have that \( rank(A) \leq rank(A|b) \). Hence \( m = rank(A) \leq rank(A|b) \). On the other hand, the number of rows of \( (A|b) \) is still \( m \), hence, \( rank(A|b) \leq m \). So we have

\[
m = rank(A) \leq rank(A|b) \leq m
\]

which implies that \( rank(A|b) = m = ran(A) \). So the system always has a solution.