1. Let

\[
A = \begin{pmatrix}
1 & -2 & 0 & 1 & 0 \\
1 & -2 & -1 & 0 & -1 \\
-2 & 4 & 1 & -1 & 1 \\
3 & -6 & 3 & 6 & 0
\end{pmatrix}.
\]

(a) Find a basis for the row space of \( A \) (i.e. the space generated by the row vectors of \( A \)).

(b) Find a basis for the column space of \( A \) (i.e. the space generated by the column vectors of \( A \)).

(c) Consider the linear defined by \( T : \mathbb{R}^5 \to \mathbb{R}^4 \) defined by (i.e. \( T \) is the left multiplication transformation associated to \( A \))

\[
T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = A \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix},
\]

where \( A \) is given above. Find a basis for \( N(T) \) (the nullspace of \( T \)) and a basis for \( R(T) \) (the range of \( T \)).

Solution.

\[
\begin{pmatrix}
1 & -2 & 0 & 1 & 0 \\
1 & -2 & -1 & 0 & -1 \\
-2 & 4 & 1 & -1 & 1 \\
3 & -6 & 3 & 6 & 1
\end{pmatrix} \to \begin{pmatrix}
1 & -2 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

So the reduced echelon form is

\[
R = \begin{pmatrix}
1 & -2 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

(a) Since \( R \) is obtained from the row-reductions from \( A \), the row-space doesn’t change. So \( \{(1, -2, 0, 1, 0), (0, 0, 1, 1, 0), (0, 0, 0, 0, 1)\} \) is a basis for the row space of \( A \).
(b) \( \{(1, 1, -2, 3), (0, -1, 1, 3), (0, -1, 1, 1)\} \) is a basis for the column space of \( A \) by Theorem 3.16.

(c) \( N(T) \) is the solution space of \( Ax = 0 \), which is also the solution space of \( Rx = 0 \), i.e.

\[
\begin{align*}
    x_1 - 2x_2 + x_4 &= 0 \\
x_3 + x_4 &= 0 \\
x_5 &= 0
\end{align*}
\]

where \( x_4, x_2 \) are free variables. So a basis for \( N(T) \) is \( \{(2, 1, 0, 0, 0), (-1, 0, -1, 1, 0)\} \).

As being discussed in the class, the range \( R(T) \) is the same as the column space of \( A \) whose basis is \( \{(1, 1, -2, 3), (0, -1, 1, 3), (0, -1, 1, 1)\} \).

2. Let \( W = \text{span}\{v_1, v_2, v_3, v_4\} \) be a subspace of \( \mathbb{R}^4 \) where \( v_1 = (1, 2, 1, 2), v_2 = (3, 1, 2, 0), v_3 = (1, -1, -1, -1), v_4 = (1, 0, 2, -1) \). Find a basis for the space \( W \).

Solution 2:

\[
\begin{pmatrix}
1 & 3 & 1 & 1 \\
2 & 1 & -1 & 0 \\
1 & 2 & -1 & 2 \\
2 & 0 & -1 & -1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

A basis for \( W \) is \( \{(1, 2, 1, 2), (3, 1, 2, 0), (1, -1, -1, -1)\} \).

3. Find a basis for the solution space of \( AX = 0 \) where

\[
A = \begin{pmatrix}
2 & -1 & 4/3 & -4 & 0 \\
1 & 0 & 2/3 & 0 & -1 \\
9 & -3 & 6 & -3 & -3
\end{pmatrix}.
\]

Solution 3:

\[
\begin{pmatrix}
2 & -1 & 4/3 & -4 & 0 \\
1 & 0 & 2/3 & 0 & -1 \\
9 & -3 & 6 & -3 & -3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 2/3 & 0 & -1 \\
0 & 1 & 0 & 4 & -2 \\
0 & 0 & 0 & 9 & 0
\end{pmatrix}
\]

A basis is \( \{(-2/3, 0, 1, 0, 0), (1, 2, 0, 0, 1)\} \)
4. Suppose \( A \) has row reduced form \( R \),

\[
A = \begin{bmatrix}
1 & 2 & 1 & b \\
2 & a & 1 & 8 \\
\star & \star & \star & \star
\end{bmatrix}, \\
R = \begin{bmatrix}
1 & 2 & 0 & 3 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

(a) What are the numbers \( a \) and \( b \)?

(b) What can you say about the third row of \( A \)?

(c) Describe the solution space of \( AX = 0 \).

**Solution:** (a) Since the system \( Ax = 0 \) is equivalent to \( Rx = 0 \), the (linear) relations between the column vectors of \( A \) is the same as the (linear) relations between the column vectors of \( R \). Notice from \( R \), we have column 2=2 times column 1, column 4=3 times column 1+2 times column 3, hence \( a = 2 \), \( b = 5 \).

(b) \( a_{32} = 2a_{31}, a_{34} = 3a_{31} + 2a_{33} \).

(c) The basis of \( N(A) \) is

\[
\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}.
\]

5. Problem 2 (a) on Page 208.

**Solution.** The determinant is 30


**Solution.** Omitted.