1. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x + y, 0, 2x - y)$.
   (a) Prove that $T$ is linear.
   (b) Find a basis for $N(T)$, the null space of $T$. Is $T$ invertible?
   (c) Find a basis for $R(T)$, the range space of $T$.
   (d) Verify the dimension theorem.
2. Prove that if $T : V \to W$ is linear, then $T(0) = 0$.
3. Let $T : P_1(\mathbb{R}) \to P_1(\mathbb{R})$ be a linear transformation defined by $T(p(x)) = p'(x)$. Let $\beta = \{1, x\}$ and $\beta' = \{1 + x, 1 - x\}$. Find $[T]_\beta$ and $[T]_{\beta'}$. Verify that $[T]_{\beta'} = Q^{-1}[T]_\beta Q$ where $Q$ is the change of coordinate matrix.
4. Let
   \[
   A = \begin{pmatrix}
   1 & 2 & 1 \\
   1 & 0 & 1 \\
   1 & 1 & 2
   \end{pmatrix}.
   \]
   (a) Find the inverse of $A$.
   (b) Write $A^{-1}$ as a product of elementary matrices.
5. Let \( T : P_2(\mathbb{R}) \to M_{2 \times 2}(\mathbb{R}) \) be the transformation given by
\[
T(f(x)) = \begin{pmatrix}
  f'(0) & 2f(1) \\
  0 & f'''(3)
\end{pmatrix}.
\]

(a) Prove that \( T \) is linear.

(b) Find the matrix of \( T \) with respect to the standard bases for \( P_2(\mathbb{R}) \) and \( M_{2 \times 2}(\mathbb{R}) \).

If \( f(x) = 3x - 2 \), compute \( T(f(x)) \) directly and using the matrix obtained in (b).

6. Find the rank of the following matrix:

\[
\begin{align*}
(a) & \begin{pmatrix}
  1 & 1 & 0 \\
  0 & 1 & 1 \\
  1 & 1 & 0
\end{pmatrix}, &
(b) & \begin{pmatrix}
  1 & 1 & 0 \\
  2 & 1 & 1 \\
  1 & 1 & 1
\end{pmatrix}.
\end{align*}
\]

7. Let \( T : P_2(\mathbb{R}) \to P_2(\mathbb{R}) \) defined by \( T(f(x)) = f(x) + f'(x) + f''(x) \).

(a) Prove that \( T \) is a linear transformation.

(b) Let \( \beta = \{1, x, x^2\} \) be the standard basis for \( P_2(\mathbb{R}) \). Find \([T]_\beta\), the matrix representation of \( T \) in the ordered bases of \( \beta \).

(c) For the matrix you obtained in (b), find its inverse.

(d) Is \( T \) invertible? If so, find \( T^{-1} \) i.e. find \( T^{-1}(a+bx+cx^2) \) for \( a, b, c \in \mathbb{R} \).