Warning: This is only a partial review. You still need to read the textbook to see all the topics we have covered. Read key to HWs and do more practices using exercises on the textbook.

1. How to use the Gauss elimination process to get the reduced echelon form?

Practice Problem: Let
\[
A = \begin{pmatrix}
1 & -2 & -1 & 1 \\
2 & -3 & 1 & 6 \\
3 & -5 & 0 & 7 \\
1 & 0 & 5 & 9
\end{pmatrix}.
\]

(a) Find the row-reduced echelon matrix \( R \) of \( A \).

(b) Find the elementary matrices \( E_1, \ldots, E_k \) such that
\[
E_k \cdots E_1 A = R.
\]

2. Solving homogeneous system of linear equations \( Ax = 0 \), where \( A \) is a \( m \times n \) matrix:

- Its solution space is a vector space, which is the same as the nullspace of \( L_A \). The dimension of the solution space is \( n - \text{rank}(A) \).

- Solutions \( x \) (complete solutions) of \( Ax = 0 \) can be written as a linear combinations of the vectors in a basis of the solution space, i.e. \( x = c_1 x_1 + \cdots + c_k x_k \), where \( \{x_1, \ldots, x_k\} \) is a basis for the solution space, and \( k = n - \text{rank}(A) \).

- \( Ax = 0 \) is equivalent to \( Rx = 0 \) where \( R \) is the reduced echelon form of \( A \). Hence a basis for the solution space \( \{x_1, \ldots, x_k\} \) can be found by looking at \( R \).

- Practice problem (see HW#9): Suppose \( A \) has row reduced form \( R \),
\[
A = \begin{bmatrix}
1 & 2 & 1 & b \\
2 & a & 1 & 8 \\
* & * & * & *
\end{bmatrix}, \quad R = \begin{bmatrix}
1 & 2 & 0 & 3 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

Describe the solution space of \( AX = 0 \).
3. **Solving non-homogeneous system of linear equations** $Ax = b$, where $A$ is a $m \times n$ matrix:

- Although the solution set $K$ is **not** a solution space, we do have

$$K = \{s\} + K_{hom}$$

where $K_{hom}$ is a solution space of the corresponding homogeneous system $Ax = 0$.

- Hence the solutions $x$ (complete solutions) can be written as

$$x = s + c_1x_1 + \cdots + c_kx_k,$$

where $s$ is a particular solution of $Ax = b$, and $x_1, \ldots, x_k$ is a basis for the solution space of the **corresponding homogeneous system** $Ax = 0$.

**Practice problems:** See HW#8 problem 2 and problem 3.

4. **For what kind $b$ that the system $Ax = b$ has a solution?**

Two theorems answer this problem: **Result 1.** (see Problem 9 on P. 181): The system $Ax = b$ has a solution if and only if $b \in R(L_A)$, i.e. $b$ is in the range of $L_A$ (see below on how to determine a basis for $R(L_A)$). **Result 2. Theorem 3.11 on P. 174:** The system $Ax = b$ has a solution if and only if $\text{rank}(A) = \text{rank}(A|b)$, where $(A|b)$ is the **augmented matrix**. In other words (see Problem 3 on Page 196), if $(A'|b')$ is the reduced echelon form of $(A|b)$, then The system $Ax = b$ has a solution if and only if $(A'|b')$ contains no rows in which the only nonzero entry lies in the last column.
Parctice problem (see HW#9): See Problem 4 (a), (b), (c), on Page 196. Also the following problem:

Let
\[ A = \begin{pmatrix} 3 & 3 & 1 \\ 3 & 5 & 1 \\ -3 & 3 & -1 \end{pmatrix}. \]

(a) Determine the condition(s) for the triples \( y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \) such that the above system \( Ax = y \) has a solution.

(b) For \( y = -1, y_2 = 0, y_3 = 4 \) does the system have solutions? If so, find the complete solutions.

5. Row space, column space, the nullspace of \( L_A \), the range space of \( L_A \):

You need know how to find a basis for each space, and why \( R(L_A) \) is the same as the column space of \( A \).

Parctice problem (see HW#9): Let
\[ A = \begin{pmatrix} 1 & -2 & 0 & 1 & 0 \\ 1 & -2 & -1 & 0 & -1 \\ -2 & 4 & 1 & -1 & 1 \\ 3 & -6 & 3 & 6 & 0 \end{pmatrix}. \]

(a) Find a basis for the row space of \( A \) (i.e. the space generated by the row vectors of \( A \)).

(b) Find a basis for the column space of \( A \) (i.e. the space generated by the column vectors of \( A \)).

(c) Consider the linear defined by \( T : \mathbb{R}^5 \to \mathbb{R}^4 \) defined by (i.e. \( T \) is the left multiplication transformation associated to \( A \))
\[ T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}, \]
where $A$ is given above. Find a basis for $N(T)$ (the nullspace of $T$) and a basis for $R(T)$ (the range of $T$). Why $R(T)$ is the column space of $A$?

6. Using Gauss elimination process and echelon form to find a basis for $\text{Span}S$, and on how to extend an independent set to a basis for the given space:

Practice problem: See example 3, and example 4. Also:

1. (a) Let $W = \text{span}\{v_1, v_2, v_3, v_4\}$ be a subspace of $\mathbb{R}^4$ where $v_1 = (1, 2, 1, 2), v_2 = (3, 1, 2, 0), v_3 = (1, -1, -1, -1), v_4 = (1, 0, 2, -1)$. Find a basis for the space $W$.

(b) Extend the set (the basis for $W$) you obtained in (a) to a basis of $\mathbb{R}^4$.

7. How to get the information about the matrix $A$ from its echelon form $R$ of $A$, see important theorem 3.16 on Page 191:

Practice problem (see HW9): Suppose $A$ has row reduced form $R$,

$$
A = \begin{bmatrix}
1 & 2 & 1 & b \\
2 & a & 1 & 8 \\
\star & \star & \star & \star
\end{bmatrix}, \quad R = \begin{bmatrix}
1 & 2 & 0 & 3 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0
\end{bmatrix}.
$$

(a) Which columns of $A$ for a basis of the column space of $A$?

(b) What are the numbers $a$ and $b$?

(c) What can you say about the third row of $A$?

(d) Describe the solution space of $AX = 0$.

8. Determinant:

Important properties for Determinant:

- If $B$ is obtained by interchanging two rows (columns) of $A$, then $\det(B) = -\det(A)$.

- If $B$ is obtained by multiplying a two (column) of $A$ by a real number $c$, then $\det(B) = c\det(A)$.
• If $B$ is obtained by adding a multiple a two (column) of $A$ to another row (column), then $\det(B) = \det(A)$.

• $\det(AB) = \det A \det B$.

• $\det(A^t) = \det(A)$

• An $n \times n$ matrix $A$ is invertible if and only if $\det(A) \neq 0$. Furthermore, if $A$ is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

9. Tow methods on computing $\det A$

   **Practice problems:** Page 321, Problem 5-12, Problem 13-22 on Page 222.

10. **Cramer’s rule:** Consider $Ax = 0$ where $A$ is a square matrix. If $\det A \neq 0$, then the system has the unique solution as below:

\[
x_k = \frac{\det(M_k)}{\det(A)}.
\]

   **Practice problems:** Example 1 on Page 225, Problem 2-7 on Page 228.