Chapter 3: The Real Numbers

1. Overview

In one sense real analysis is just doing calculus all over again, only this time we prove everything. But in another larger sense this class is much more than that. It’s about setting up a system to analyze things like calculus thoroughly and rigorously so that we can move beyond calculus.

Our system is built up on the axiomatic assumptions (or definitions) on the real numbers.

So, what is a real number? In mathematics, the real numbers may be described informally as numbers that can be given by an infinite decimal representation, such as 2.4871773339. The real numbers include both rational numbers, such as 42 and -23/129, and irrational numbers, such as π and √2, and can be represented as points on an infinitely long number line.

A more rigorous definition of the real numbers was one of the most important developments of 19th century mathematics. In this book, we use an axiomatic definition (i.e. we assume these properties automatically hold) of the real numbers as the unique complete Archimedean ordered field, i.e. we assume that \( \mathbb{R} \) has the following mathematical structures (click here to see more):

(a) (Archimedean fields): There are two operations “+” and “·” on \( \mathbb{R} \), i.e. we can do addition and multiplication for any two real numbers;

(b)(ordered): There is also an order or relation “<” on \( \mathbb{R} \), i.e. we can compare two real numbers to see which one is bigger;

(c) (complete): Every nonempty subset \( S \) of \( \mathbb{R} \) that is bounded above has a least upper bound. That is, sup \( S \) exists and is a real number (i.e. the set \( \mathbb{R} \) is big enough, it contains sup \( S \) for every subset \( S \)).

Note that axioms are some starting assumptions from which other statements are logically derived. Unlike theorems, axioms cannot be derived by principles of deduction nor demonstrable by formal proofs, simply because they are starting assumptions and there is nothing else they logically follow from (otherwise they would be called theorems). Based on these axioms, more theorems are derived in Section 12.
Using "<", we can define $|x|$, the absolute values of $x$. Using the absolute value, we have the concept of the distance of two real numbers. The distance concept allows us to define the neighborhood (see section 13, P. 129). Then we can introduce the concepts of interior point, boundary point, open set, closed set, etc.. (see Section 13: Topology of the reals). All these concepts have something to do with the distance, which describes how close two points are. These concepts will be used in the study of limit, continuity, etc..

Finally, we introduce the concept of compact set (see section 14). A set $S$ in $\mathbb{R}$ is compact if and only if it is closed and bounded. The most important property for a compact set $S$ is that every open cover (usually it contains infinitely many open sets) has a finite subcover. This important property allows us to pass from infinitely many to finitely many (this trick will be used quite often in the proofs of this course).

### Section 10: Induction

When a statement involves natural numbers (i.e. non-negative integers), you usually use the method of induction (click here to see more) to prove it. The procedure goes as follows: Step 1: Verify the statement holds for $n = 1$ (base step); Step 2: Assume that, for each natural number $n$, the statement holds for $n$ (Induction hypothesis, or IH), try to verify the statement also holds for $n + 1$.

**Example.** Prove that

$$\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}.$$ 

**Proof:** We use induction on $n$.

**Base Step:** $n = 1$. Then

$$\frac{1}{2!} = \frac{1}{2} = 1 - \frac{1}{1+1}.$$ 

**Inductive Step:** Assume that $n \geq 1$ and that (here (IH) means the “induction hypothesis”)

$$(\text{IH}) \quad \frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}.$$ 

Then

$$\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n+1}{(n+2)!} = \frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} + \frac{n+1}{(n+2)!}$$
\[ 1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!} \quad \text{by (IH)} \]
\[ = 1 - \frac{1}{(n+1)!} \left( 1 - \frac{n+1}{n+2} \right) \]
\[ = 1 - \frac{1}{(n+1)!} \left( \frac{1}{n+2} \right) \]
\[ = 1 - \frac{1}{(n+2)!}. \]

Hence, by induction, we have proved that
\[ \frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}. \]

**Section 11, Ordered Fields**

This section explains that is a **field** and what an **order** is.

When we say that \( \mathbb{R} \) is a **field** (click here to see more), we mean that there are two operations “+” (addition) and “\( \cdot \)” (multiplication) on \( \mathbb{R} \) which satisfy 11 properties (see Page 108-109, A1-DL).

By the **ordered field** (click here to see more), we mean that, in addition to two operations “+” (addition) and “\( \cdot \)” (multiplication) on \( \mathbb{R} \), there is also an **order** or relation “<” which satisfy Q1-Q4 properties on Page 109. We assume these properties at the beginning which are called **axioms**, i.e. we admit them automatically without proofs.

Based on these **axioms**, this chapter continues derives **Theorems**. see, for example, Theorem 11.1, Theorem 11.7. Read **Practice 11.2-11.6** and **Example 11.5** carefully to see how to use these axioms and theorems to do the proofs (helpful in doing HWs).

On Page 113, it introduces the concept of **absolute value** by using the order “\( \cdot \)”. Then it derives the properties of the absolute value. The concept of absolute value is essential in section 13.

Note that in doing HW, **what you can use freely** are those assumptions in the axiom (of ordered field) plus the theorems or results which you have already proved. See the following example.
11.3(a): Show that $-(-x) = x$.

Proof:

$(-x) + (-(-x)) = 0 = x + (-x)$ \hspace{1cm} \text{by Axiom A5}

$(-x) + (-(-x)) = (-x) + x$ \hspace{1cm} \text{by Axiom A2}

$x + ((-x) + (-(-x))) = x + ((-x) + x)$ \hspace{1cm} \text{by Axiom A1}

$(x + (-x)) + (-(-x)) = (x + (-x)) + x$ \hspace{1cm} \text{by Axiom A3}

$0 + (-(-x)) = 0 + x$ \hspace{1cm} \text{by Axiom A5}

$(-(-x)) + 0 = x + 0$ \hspace{1cm} \text{by Axiom A2}

$-(-x) = x$ \hspace{1cm} \text{by Axiom A4}.
Section 12: The Completeness Axiom

Note that the axioms about “+”, “·”, and “<” does not fully characterize \( \mathbb{R} \). We need another important axiom on \( \mathbb{R} \) in section 12, called completeness axiom as follows: Every nonempty subset \( S \) of \( \mathbb{R} \) that is bounded above has a least upper bound. That is, \( \sup S \) exists and is a real number. This means that the set \( \mathbb{R} \) is big enough, it contains all \( \sup S \) for every subset \( S \). So the full characterization for \( \mathbb{R} \) is that \( \mathbb{R} \) is a complete ordered field.

First of all, in this section, you need to make clear what are the meanings of upper bound, least upper bound (or supremum), greatest lower bound (or infimum), maximum (the achieved upper bound, i.e. it is the (least) upper bound which is also in the set \( S \)), and minimum (also achieved).

Here are some highlights:

- 1. The axiom of \( \mathbb{R} \) is \textbf{complete} is essential, i.e. for every subset \( S \) of \( \mathbb{R} \) which is bounded above, \( \sup S \) always exists!!!. Note, \( \mathbb{Q} \), the set of all rational numbers, however, does not have such property, for example, let \( S = \{ q \in \mathbb{Q} \mid 0 \leq q \leq \sqrt{2} \} \), then \( \sup S \) does not exist in \( \mathbb{Q} \). In fact, \( \sup S = \sqrt{2} \) is not a rational number. This shows that \( \mathbb{Q} \) is not bigger enough!, you now see why we need to study \( \mathbb{R} \), rather than \( \mathbb{Q} \).

- 2. Although \( \sup S \) always exists, for all subset \( S \) of \( \mathbb{R} \) which is bounded above, \( \sup S \) may not be in the set \( S \). For example, let \( S = \{ x \in \mathbb{R} \mid 0 \leq x < \sqrt{2} \} \), then \( \sup S = \sqrt{2} \), however, \( \sqrt{2} \notin S \). If \( \sup S \) is in \( S \), then we call it maximum of \( S \), denoted by \( \max S \).

- 3. If \( S \) is a finite set, then \( \max S \) and \( \min S \) always exist. This property is frequently use in section 14 (the concept of compactness allows us pass from infiniteness to finiteness).

The last part of the section deals with the density of rational numbers in the real numbers, i.e. for every two real numbers \( x < y \), there is a rational number \( r \) such that \( x < r < y \).

Section 13: The Topology of the Reals

By the topology of \( \mathbb{R} \), we mean the collection (or the set) of all open subset of \( \mathbb{R} \). Hence, you see that the main purpose of this section is to introduce the concept
of open set (of course, as well as other notions: interior point, boundary point, closed set, open set, accumulation point of a set $S$, isolated point of $S$, the closure of $S$, etc.).

The approach is to use the distance (or absolute value). First, it introduce the concept of neighborhood of a point $x \in \mathbb{R}$ (denoted by $N(x, \epsilon)$) see (page 129)(see also the deleted neighborhood). It is the most convenient concept (think about what does your neighbor mean: it means someone lives within your distance). It then introduces interior point (i.e. if you are surrounded by your neighbors, then you are the interior point), boundary point, ....

Read the examples and try to do exercises to see (1): how to determine whether it is open, close, interior points, etc., (2) How to write rigorous proofs once you conclude your answer.

One of the most important properties is: (a) The union of any collection (can be infinite) of open sets is an open set, (b) The intersection of any finite collection of open sets is an open set. The topology of $\mathbb{R}$ refers to the collection of all open sets of $\mathbb{R}$ (i.e., in the abstract setting: A topological space $X$ is a set with a collection of subsets, denoted $T$ (elements in $T$ are called open sets), such that (a) and (b) holds).

For the closed set, we have the following properties: (a) The finite union of any collection of closed sets is a closed set, (b) The intersection of any collection (can be infinite) of closed sets is closed set.

Try to use the terms we introduced to do some proofs.

**11.9(a): Prove that an accumulation point of a set $S$ is either an interior point of $S$ or a boundary point of $S$.**

**Proof:** Let $x \in \mathbb{R}$ be an accumulation point of $S$, and assume that $x$ is not an interior point of $S$(otherwise we are done). We need to show that $x$ is boundary point of $S$. So let $\epsilon > 0$; we need to show that $N(x; \epsilon) \cap S \neq \emptyset$ and $N(x; \epsilon) \cap (\mathbb{R} \setminus S) \neq \emptyset$. On the one hand, since $x$ is an accumulation point of $S$, we know by definition that $N^*(x; \epsilon) \cap S \neq \emptyset$; since $N^*(x; \epsilon) \subset N(x; \epsilon)$, it follows that $N(x; \epsilon) \cap S \neq \emptyset$. On the other hand, since $x$ is not an interior point of $S$, we cannot have $N(x; \epsilon) \subset S$; that is $N(x; \epsilon) \cap (\mathbb{R} \setminus S) \neq \emptyset$. This proves our claim.

**Section 14: Compact Sets**
A set $S$ is **compact** (click here to see more), if and only if every open cover $\mathcal{F}$ (usually it contains infinitely many open sets) has a finite subcover $\mathcal{G}$. This **important property** allows us to pass from infinitely many to finitely many. Almost all proofs of this section uses this important property.

The set $(0, 1]$ is not compact (it is not closed), for example, the $\{(1/n, 1 + (1/n))\}$ is an open cover for $(0, 1]$ but you can not find a finite sub-cover.

As I mentioned, the important property of compactness allows us to pass from infinitely many to finitely many. In practice, how to do it? **Step 1**, construct an open cover, **step 2** Use the compactness to pass from infinitely cover to a finitely subcover, and see how to derive your conclusion by using the finiteness property. Note the proof is often combined with the method of proof by contradiction (see the proof of Theorem 14.7 on P. 142).

As a warm up, the proof of the statement that *If a set $S$ in $\mathbb{R}$ is compact, then it is bounded* goes as follows: We can cover the set $S$ by $I_n = (-n, n)$, i.e. $S \subset \bigcup_{n=1}^{\infty} I_n$. Since $S$ is compact, this open cover $\{I_n\}$ has a finite subcover, i.e. there exist finite many integers $n_1, \ldots, n_k$ such that

$$S \subset (I_{n_1} \cup \cdots \cup I_{n_k}).$$

Let $m = \max\{n_1, \ldots, n_k\}$ (note for a finite set, a maximum always exists!!!). Hence $S \subset (-m, m)$. SO $S$ is bounded. Note, you should learn from this proof how to: **Step 1**, construct an open cover, **step 2** Use the compactness to pass from infinitely cover to a finitely subcover, and see how to derive your conclusion by using the finiteness property.

Another warm up: try to prove the statement that *If a set $S$ in $\mathbb{R}$ is compact, then it is closed.*

Of course, Heine-Borel (see Theorem 14.5) (click here to see more), gives an important characterization: *A set $S$ in $\mathbb{R}$ is compact if and only if it is closed and bounded.*