bf MATH 6322, Complex Analysis Fall 2011, Key
to HW#6

October 31, 2011

Additional Problems.
1. Let $a \in D(0,1)$, and define the Möbius transform:
\[
\phi_a(z) = \frac{z - a}{1 - \bar{a}z}
\]

Prove that
a). $\phi_a$ maps the unit disc into unit disc, and $\phi_a$ is holomorphic on $D(0,1)$.
b). Prove that $\phi_a : D(0,1) \to (0,1)$ is one-to-one and onto and its inverse is $\phi_{-a}$.
c). Use a), b) and Schwartz Lemma to prove the Schwartz-Pick theorem.

Proof.
a). It is clear that $\phi_a(z) = \frac{z - a}{1 - \bar{a}z}$ is holomorphic on $D(0,1)$, since $1 - \bar{a}z$ has no zeros on $D(0,1)$.

To show that $\phi_a$ maps the unit disc into unit disc, for $z \in D(0,1)$, we compare $|z - a|^2$ with $|1 - \bar{a}z|^2$, and have
\[
|z - a|^2 - |1 - \bar{a}z|^2 = |z|^2 - 2\Re(\bar{a}z) + |a|^2 - (1 - 2\Re(\bar{a}z) + |a|^2|z|^2)
= |z|^2 + |a|^2 - 1 - |a|^2|z|^2
= (1 - |z|^2)(|a|^2 - 1) < 0
\]
since $|z| < 1$ and $|a| < 1$. Hence, $|z - a|^2 - |1 - \bar{a}z|^2$, which implies $\left| \frac{z - a}{1 - \bar{a}z} \right| < 1$, i.e.
\[
|\phi_a(z)| < 1, \quad \text{for each } z \in D(0,1).
\]
b). For each $w \in D(0,1)$, we solve $\phi_a(z) = w$, i.e.
\[
\frac{z - a}{1 - \bar{a}z} = w,
\]
to get
\[ z = \frac{w + a}{1 + \bar{a}w}. \]

Using (a) above, we have that \( |z| < 1 \). Since \( z \) is unique for each \( w \in D(0, 1) \), \( \phi_a \) is one-to-one and onto, and the inverse of \( \phi_a(z) \) is
\[ \phi_{-a}(w) = \frac{w + a}{1 + \bar{a}w}, \quad w \in D(0, 1). \]

c). Assume \( f(a_1) = b_1 \) and \( f(a_2) = b_2 \), then we take
\[ h(z) = \phi_{b_1} \circ f \circ \phi_{-a_1}(z) \]
So \( h \) is a holomorphic function with \( h(0) = 0 \) and \( |h(z)| < 1 \) for all \( z \in D(0, 1) \), and using the Schwartz Lemma gives us
\[ |h(z)| = |\phi_{b_1} \circ f \circ \phi_{-a_1}(z)| \leq |z| \]
Put \( z = \phi_{a_1}(w) \), we then have
\[ |\phi_{b_1} \circ f(w)| \leq |\phi_{a_1}(w)| \]
Put \( w = a_2 \), and we obtain
\[ |\phi_{b_1}(b_2)| \leq |\phi_{a_1}(a_2)| \]
This implies that, by the definition of \( \phi_{a_1} \) and \( \phi_{b_1} \),
\[ \left| \frac{b_2 - b_1}{1 - b_1b_2} \right| \leq \left| \frac{a_2 - a_1}{1 - \bar{a}_1a_2} \right| \]
And this completes the proof of the Schwartz-Pick Lemma.

By letting \( a_1 = a, a_2 = z \), using above, we get
\[ \left| \frac{f(z) - f(a)}{z - a} \right| \leq \left| \frac{1 - \bar{a}f(a)}{1 - az} \right|. \]
By letting \( z \to a \), we get
\[ |f'(a)| \leq \frac{1 - |f(a)|^2}{1 - |a|^2}. \]
\[ \square \]

2. Let \( f \) be holomorphic on \( D(0, 1) \subset U \) and \( |f(z)| \leq 1 \) on \( D(0, 1) \), then prove that for all \( z \in D(0, 1) \),
\[ (1 - |z|^2)|f'(z)| \leq 1 \]
Proof.
By the Schwartz-Pick Lemma, for each $z \in D(0, 1)$, we have that

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}$$

Then, as $1 - |f(z)|^2 \leq 1$, it directly follows that

$$(1 - |z|^2)|f'(z)| \leq 1.$$ 

\[\square\]

**Chapter 4.**

5. Classify the following singularities.

(a). $\frac{1}{z}$

(b). $\sin \frac{1}{z}$

(e). $\frac{\sin z}{z}$

**Proof.**

(a). Pole: $z = 0$.

(b). Because on $\mathbb{C}$, $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$, hence, on $0 < |z| < \infty$,

$$\sin(1/z) = z^{-1} \frac{z^{-3}}{3!} + \frac{z^{-5}}{5!} - \cdots$$

So $z = 0$ is an essential singularity:

Alternative argument: This is because that when we take $\{z_n = \frac{1}{n\pi} : n \in \mathbb{N}\}$ to approach 0, then $\sin \frac{1}{z_n} = 0$ for all $n \in \mathbb{N}$; however, if we take $\{z_n = \frac{i}{n\pi} : n \in \mathbb{N}\}$ to approach 0, then $\sin \frac{1}{z_n} \to \infty$ when $n \to \infty$.

(e). Removable singularity: $z = 0$.

This is because $\sin z$ has the power series expansion as $z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$ around zero, hence $\frac{\sin z}{z}$ behaves as $1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$ around zero, i.e. $\lim_{z \to 0} \frac{\sin z}{z} = 1$, which implies that $\frac{\sin z}{z}$ is bounded near zero.

\[\square\]

13. Calculate the annulus of convergence (including any boundary points) for the following Laurent series:

(a). $\sum_{j=-\infty}^{\infty} 2^{-j} z^j$

(b). $\sum_{j=0}^{\infty} 4^{-j} z^j + \sum_{j=-1}^{-\infty} 3^j z^j$

(f). $\sum_{j=-20}^{\infty} j^2 z^j$
Solution.

(a). Write $\sum_{j=-\infty}^{\infty} 2^{-j}z^j$ as

$$\sum_{j=-\infty}^{\infty} 2^{-j}z^j = \sum_{j=0}^{\infty} 2^{-j}z^j + \sum_{j=-\infty}^{-1} 2^{-j}z^j$$

The disk of convergence for $\sum_{j=0}^{\infty} 2^{-j}z^j$ is $|z| < 2$. However, for $\sum_{j=-\infty}^{-1} 2^{-j}z^j = \sum_{j=1}^{\infty} 2^jz^{-j}$, we know it converges on $|z| > 2$, diverges on $|z| < 2$. Thus the annulus of convergence (including any boundary points) for $\sum_{j=-\infty}^{\infty} 2^{-j}z^j$ is empty.

(b). First, consider $\sum_{j=0}^{\infty} 4^{-j}z^j$, its disk of convergence is, by the root test, $|z| < 4$.

Next, for $\sum_{j=-\infty}^{-1} 3^{-j}z^j = \sum_{j=1}^{\infty} 3^{-j}z^{-j}$, we know that it is, by the root test, converges on $|z^{-1}| < 3$. i.e. $|z| > 1/3$, So the annulus of convergence is $1/3 < |z| < 4$.

(f). There are only finitely many negative exponent terms of this Laurent series, and notice that $\sum_{j=-20}^{\infty} j^2z^j$ is defined for all $z \neq 0$. Next for $\sum_{j=1}^{\infty} j^2z^j$, it converges, by the root test on $|z| < 1$. Hence, the annulus of convergence is $0 < |z| < 1$.


27. Calculate the first four terms of the Laurent expansion of the given function about the given point. In each case, specify the annulus of convergence of the expansion.

(a). $f(z) = \csc z$ about $P = 0$

(b). $f(z) = z/(z+1)^3$ about $P = -1$

(c). $f(z) = z/[(z-1)(z-3)(z-5)]$ about $P = 1$

(h). $f(z) = e^z/z^3$ about $P = 0$

Proof.

(a). Write $f(z) = \csc z$ as $f(z) = \frac{1}{\sin z}$. From $\lim_{z \to 0} \frac{\sin z}{z} = 1$, we know that $P = 0$ is a pole of order 1. Thus, from the formula in Proposition 4.4.1 (Page 119), $a_{-n} = 0$ for all $n > 2$, and $a_{-1} = z f(z)|_{z=0} = 1 = \lim_{z \to 0} \frac{z}{\sin z} = 1$, $a_0 = \frac{d}{dz}(zf(z))|_{z=0} = 0$, $a_1 = \frac{d^2}{dz^2}(zf(z))|_{z=0} = 1/6$ and $a_2 = \frac{d^3}{dz^3}(zf(z))|_{z=0} = 0$.

Alternatively, you can use the integral formula for $a_n$ (see Proposition 4.3.3 (Page 116)) to compute the first four terms.
(b). Write \( f(z) = \frac{z}{(z + 1)^3} \) as
\[
    f(z) = \frac{z}{(z + 1)^3} = \frac{z - (-1) - 1}{(z - (-1))^3}.
\]
Then, we know that the only zero for \((z - (-1))^3\) is \(z = -1\), so the annulus of convergence for \(f(z) = \frac{z}{(z + 1)^3}\) is \(0 < |z + 1| < \infty\).
Moreover, we have
\[
    f(z) = \frac{z - (-1) - 1}{(z - (-1))^3} = \frac{1}{(z - (-1))^2} - \frac{1}{(z - (-1))^3},
\]
i.e. \(a_{-2} = 1, a_{-3} = -1\) and \(a_n = 0\) for all \(n \neq -2, -3\).

(c). Obviously \(z = 1\) is a pole of \(f(z)\) with order 1. Thus, form the formula on book, \(a_{-n} = 0\) for all \(n > 2\), and \(a_{-1} = (z - 1)f(z)|_{z=1} = 1/8, a_0 = \frac{\partial}{\partial z^2}((z - 1)f(z))|_{z=1} = 7/32, a_1 = \frac{\partial^2}{\partial z^2}((z - 1)f(z))|_{z=1} = 19/128\) and \(a_2 = \frac{\partial^3}{\partial z^3}((z - 1)f(z))|_{z=1} = 43/512\).

(h). 
\[
    f(z) = \frac{e^z}{z^3} = \frac{1}{z^3} \left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots\right)
    = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + \frac{1}{6} + \cdots
\]
so \(a_{-3} = a_{-2} = 1, a_{-1} = 1/2, a_0 = 1/6\).

\(\square\)