Key to Homework 9, Thanks to Da Zheng for providing the tex-file

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1. Page 204, #20

Let \( \{ f_\alpha \} \) be a normal family of holomorphic functions on a domain \( U \). Prove that \( \{ f'_\alpha \} \) is a normal family.

**Proof.**

Pick any sequence \( \{ f'_n \} \subseteq \{ f'_\alpha \} \), then by the given condition, for the sequence \( \{ f_n \} \), there is a subsequence which converges uniformly on compact sets of \( U \), say \( \{ f_{n_k} \} \). By Corollary 3.5.2, \( \{ f'_{n_k} \} \) converges uniformly on compact subset of \( U \). Hence, for any sequence of \( \{ f'_n \} \), we can find a subsequence which converges uniformly on compact subsets, which implies that \( \{ f'_\alpha \} \) is a normal family by definition.

\[ \square \]

2. Page 205, #24

Let \( \Omega \subseteq \mathbb{C} \) be a bounded domain and let \( \{ f_j \} \) be a sequence of holomorphic functions on \( \Omega \). Assume that

\[ \int_{\Omega} |f_j(z)|^2 \, dxdy < C < \infty \]

where \( C \) does not depend on \( j \). Prove that \( \{ f_j \} \) is a normal family.

**Proof.** First we note that, as the hint given in the problem, you can use the hint in Problem 8, chapter 4 (Page 146), which you need to show that, if \( f \) is holomorphic on \( D(Q, \epsilon) \), then

\[ |F(Q)|^2 \leq \frac{1}{\pi \epsilon^2} \int_{D(Q, \epsilon)} |F(z)|^2 \, dxdy. \]

We will follow the hint given on Page 146 in proving the above. Below is the proof of problem #24
Solution: We will show that \( \{f_j\} \) is uniformly bounded on every compact subsets of \( \Omega \), then the Montel theorem in the book will imply that \( \{f_j\} \) is normal on \( \Omega \). To do so, let \( K \subset \Omega \) be compact. By the Lebesgue number lemma (see Munkers: Topology P. 175-176, or a better proof can be found at http://mathblather.blogspot.com/2011/07/lebesgue-number-lemma-and-corollary.html), there exists \( r_K > 0 \) such that for each \( z \in K \), \( D(z, r_K) \subset \Omega \). Now fix \( Q \in K \).

by the Cauchy integral formula, for every \( 0 \leq r \leq r_K \),

\[
f_j^2(Q) = \frac{1}{2\pi i} \oint_{\partial D(Q,r)} \frac{f_j^2(\zeta)}{\zeta - Q} d\zeta
\]

So, if we parameterize \( \partial D(Q,r) \) as \( re^{i\theta} \), where \( \theta \in [0, 2\pi] \).

\[
|f_j(Q)|^2 = \left| \frac{1}{2\pi i} \oint_{\partial D(Q,r)} \frac{f_j^2(\zeta)}{\zeta - Q} d\zeta \right|
\]

\[
\leq \frac{1}{2\pi} \oint_{\partial D(Q,r)} \left| \frac{f_j^2(\zeta)}{\zeta - Q} \right| |d\zeta|
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} |f_j(Q + re^{i\theta})|^2 r d\theta
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} |f_j(Q + re^{i\theta})|^2 d\theta
\]

Now, use the Fubini theorem (and the polar coordinates) as well as the above inequality,

\[
\int_{D(Q,r_K)} |f_j(x,y)|^2 dx dy = \int_0^{r_K} \int_0^{2\pi} |f_j(Q + re^{i\theta})|^2 rdrd\theta
\]

\[
\geq 2\pi \int_0^{r_K} |f_j(Q)|^2 rdr \geq 2\pi \frac{r_K^2}{2} |f_j(Q)|^2.
\]

Thus, by the assumption,

\[
C > \int_{\Omega} |f_j(z)|^2 dx dy \geq \int_{D(Q,r_K)} |f_j(x,y)|^2 dx dy = \pi r_K^2 |f_j(Q)|^2.
\]

Hence, for every \( Q \in K \),

\[
|f_j(Q)|^2 \leq \frac{C}{\pi r_K^2}
\]

which proves our claim. \( \square \)

3. (Marty’s Theorem)
Let $\mathcal{F}$ be a family of holomorphic functions on a region $U$ on $\mathbb{C}$. Prove that $\mathcal{F}$ is normal (in the general sense) if and only if for every compact subset $K$ of $U$, there is a constant $C_K$ such that

$$f^\#(z) \leq C_K$$

for all $z \in K$ and $f \in \mathcal{F}$, where

$$f^\#(z) := \frac{|f'(z)|}{1 + |f(z)|^2}$$

**Proof.**

"$\Rightarrow$" Suppose that we have $\mathcal{F}$ as a normal family, but it does not satisfy the Marty's Criterion, i.e. there exists a compact subset $K$, a sequence of points $z_n \in K$, and a sequence of functions $\{f_n\} \subseteq \mathcal{F}$, such that

$$f_n^\#(z_n) := \frac{|f'(z_n)|}{1 + |f(z_n)|^2} \geq n, \text{ for each } n$$

However, $\mathcal{F}$ being a normal family implies that either $\{f_n\}$ has a uniformly convergent subsequence on $K$, say $\{f_{n_k}\}$, or uniformly divergent subsequence on $K$, say $\{f_{n_l}\}$.

Next, if $\{f_{n_k}\}$ converges uniformly to $\hat{f}$, then $\{f'_{n_k}\}$ also converges uniformly to $\hat{f}'$, so

$$f_{n_k}^\#(z) = \frac{|f'_{n_k}(z)|}{1 + |f_{n_k}(z)|^2} \to \hat{f}^\#(z) = \frac{|\hat{f}'(z)|}{1 + |\hat{f}(z)|^2} \text{ uniformly}$$

which contradict the our assumption at the beginning.

If we have $\{f_{n_l}\}$ diverges uniformly on $K$, then $\{\frac{1}{f_{n_l}}\}$ converges uniformly to 0, which means

$$\left(\frac{1}{f_{n_l}}\right)^\#(z) = f_{n_l}^\#(z) = \frac{|f'_{n_l}(z)|}{1 + |f_{n_l}(z)|^2} \to 0 \text{ uniformly}$$

which again contradicts our previous assumption.

Thus, we proved that if $\mathcal{F}$ is a normal family, then it satisfies the Marty’s Criterion.

"$\Leftarrow$" There are Three approaches (may be more) in proving Marty’s theorem, the second and third are similar. The first approach is the most natural one.

**The Method I:** To deal with the $\infty$, we study the extended complex plane, i.e. $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Recall that, in HW8, we can regard $\mathbb{C} \cup \{\infty\}$ as $S^2$ through
the stereographic projection. The **spherical distance** formula proved in last HW (HW8) is that for \( z, w \in \mathbb{C} \),

\[
\sigma(z, w) = \frac{2|z - w|}{\sqrt{(1 + |z|^2)(1 + |w|^2)}}.
\]

Also you can check if \( z \in \mathbb{C} \)

\[
\sigma(z, \infty) = \frac{2}{\sqrt{(1 + |z|^2)}}.
\]

Moreover, we find that if \( f \) is holomorphic on \( U \), then

\[
f^\#(z) = \frac{|f'(z)|}{1 + |f(z)|^2} = \lim_{w \to z} \frac{\sigma(f(w), f(z))}{|w - z|}.
\]

First, you need to study the relationship of “the sequence \( \{f_n\} \) converges uniformly in the spherical distance on every compact subset \( K \subset U \)” with our convergence in the usual sense (in the Euclidean norm).

**Lemma 1:** Let \( \{f_n\} \) be a sequence of holomorphic functions on a region \( U \subset \mathbb{C} \). Then \( \{f_n\} \) converges uniformly in the spherical distance on every compact subset \( K \subset U \) if and only if that for every \( z \in U \), there is a neighborhood \( D(z, r) \subset U \) such that either \( \{f_n\} \) or \( \{\frac{1}{f_n}\} \) converges uniformly in the usual sense (in the Euclidean norm).

**Proof.** “\( \Leftarrow \)” is trivial by the Heini-Borel theorem (i.e. every compact \( K \) has a finite open sub-cover for every open covering of \( K \)).

“\( \Rightarrow \)”: Let \( f \) be the limit function of \( \{f_n\} \) (with respect to the spherical distance). Take \( z_0 \in D \). Since \( \sigma(z, w) = \sigma(1/z, 1/w) \), we only need to discuss the case that \( f(z_0) \neq \infty \). We claim that \( f \) does not take \( \infty \) in a neighborhood of \( z_0 \). Indeed, take \( \delta > 0 \) with \( \overline{D(z_0, \delta)} \subset U \). From the assumption, there is \( m \) such that for all \( z \in \overline{D(z_0, \delta)} \)

\[
\sigma(f(z), f_m(z)) < \frac{1}{6\sqrt{1 + |f(z_0)|^2}}.
\]

For such (fixed) \( m \), take \( r < \rho \) such that for \( z \in D(z_0, r) \),

\[
\sigma(f_m(z), f_m(z_0)) < \frac{1}{6\sqrt{1 + |f(z_0)|^2}}.
\]

Thus, for \( z \in D(z_0, r) \),

\[
\sigma(f(z), f(z_0)) \leq \sigma(f(z), f_m(z)) + \sigma(f_m(z), f_m(z_0)) + \sigma(f_m(z_0), f(z_0)) \leq \frac{1}{2\sqrt{1 + |f(z_0)|^2}}.
\]
Hence \( f(z) \neq \infty \) on \( D(z_0, r) \) (by using the fact that \( \sigma(z, \infty) = \frac{2}{\sqrt{1+|z|^2}} \)) and
\[
\frac{|f(z) - f(z_0)|}{\sqrt{1 + |f(z)|^2}} < \frac{1}{2}.
\]
Thus, \( f \) is bounded on \( D(z_0, r) \), say \( |f(z)| \leq M \) on \( D(z_0, r) \). This proves the claim. From the assumption again, there exists \( N > 0 \) such that for \( n > N \), on \( D(z_0, r) \),
\[
\sigma(f(z), f_n(z)) < \frac{1}{2}.
\]
Thus, on \( D(z_0, r) \),
\[
\frac{1}{\sqrt{1 + M^2}} \leq \frac{1}{2 \sqrt{1 + |f(z)|^2}} = \sigma(f(z), \infty) \leq \sigma(f(z), f_n(z)) + \sigma(f_n(z), \infty) < \frac{1}{2 \sqrt{1 + M^2}} + \sigma(f_n(z), \infty),
\]
which gives, on \( D(z_0, r) \),
\[
\sqrt{1 + |f(z)|^2} < 2 \sqrt{1 + M^2}.
\]
Therefore, we know that for \( n > N \), there are no poles on \( D(z_0, r) \) and, on \( D(z_0, r) \),
\[
|f_n(z) - f(z)| < 2 \sqrt{1 + M^2} \sigma(f(z), f_n(z)).
\]
This means that \( \{f_n\} \) converges uniformly to \( f \) in the usual sense. This proves the lemma.

**Corollary:** Let \( \{f_n\} \) be a sequence of holomorphic functions on a region \( U \subset \mathbb{C} \) (Assume \( U \) is connected), which converge uniformly in the spherical distance on every compact subset of \( U \) to a function \( f : U \to \mathbb{C} \cup \{\infty\} \). Then either \( f \) is holomorphic on \( U \) (with no poles) or \( f \equiv \infty \).

**Proof.** Assume that \( f \neq \infty \), i.e. there is \( z_0 \in U \) with \( f(z_0) \neq \infty \). We need to prove that \( f \) is holomorphic. For the lemma (and its proof) above, the limit function \( f : U \to \mathbb{C} \cup \{\infty\} \) has the following property: for every point \( z \in U \), there is a \( r > 0 \) with \( D(z, r) \subset U \) such that either \( f \) is holomorphic on \( D(z, r) \) or \( \frac{1}{f} \) is holomorphic on \( D(z, r) \). Now for any (fixed) \( z \in U \), there is line segments in \( U \), say \( L \), which connected \( z_0 \) and \( z \). By Heine-Borel again, there are finite coverings \( D(z_i, r_i), 1 \leq i \leq m \), for \( L \) with \( D(z_{i-1}, r_{i-1}) \cap D(z_i, r_i) \neq \emptyset \). Since \( f(z_0) \neq \infty \), we know that \( f \) is holomorphic on \( D(z_0, r_0) \); which, in turn, implies that \( f \) is holomorphic on \( D(z_1, r_1) \), ..., eventually, we can get \( f \) is holomorphic on \( D(z_m, r_m) \). Thus \( f \) is holomorphic at \( z \) for any \( z \in U \). This proves the Corollary.

For the above lemma and Corollary, we conclude that
**Theorem:** Let $\mathcal{F}$ be a family of holomorphic functions on a region $U$ on $\mathbb{C}$. Prove that $\mathcal{F}$ is normal (in the general sense) if and only if for every sequence $\{f_n\} \subset \mathcal{F}$, there is a subsequence such that it converges uniformly in spherical distance on every compact subset of $U$.

Once the above theorem is established, you can use the Ascoli-Arzala’s theorem (note that Ascoli-Arzala’s theorem works for any distance, so it also works for the spherical distance). Note that $S^2$ is compact, so the “uniformly bounded” condition is NOT a problem. So you only need to check that $\mathcal{F}$ is equi-continuous on every compact subset $K \subset U$. So you only need to check that $\mathcal{F}$ is equi-continuous on every compact subset $K \subset U$. This can be achieved as follows:

Recall that if $f$ is holomorphic on $U$, then

$$f^\#(z) = \frac{|f'(z)|}{1 + |f(z)|^2} = \lim_{w \to z} \frac{\sigma(f(w), f(z))}{|w - z|}. $$

This implies the following equality

$$\sigma(f(z + h), f(z)) = f^\#(z) \cdot |h| + o(|h|).$$

Then, taking any compact subset $K$ of $U$, and $z_1, z_2 \in K$ ($z_1$ and $z_2$ are close enough so that the line segment between them lies in $U$), for an arbitrary $f \in \mathcal{F}$, we set $h = (z_1 - z_2)/n$, and shall have

$$\sigma(f(z_1), f(z_2)) \leq \sum_{i=0}^{n-1} \sigma(f(z_1 + ih), f(z_1 + (i + 1)h))
= \sum_{i=0}^{n-1} (f^\#(z_1 + ih)|h| + o(|h|))
\leq C_K |z_1 - z_2| + \frac{|z_1 - z_2|}{|h|} o(|h|)$$

So letting $n \to +\infty$, we have

$$d(f(z_1), f(z_2)) \leq C_K |z_1 - z_2|, \quad \text{for each } z_1, z_2 \in K \quad (*)$$

Using the Arzela’s theorem (with the fact that $S^2$ is compact), similar to the proof of Montel’s theorem in the book, by taking a compact exhaustion, named $\{K_n\}$, of $U$ and using the “diagonal method”, we can prove that for every sequence $\{f_n\} \subset \mathcal{F}$, there is a subsequence such that it converges uniformly in spherical distance on every compact subset of $U$. By the theorem above, it means that $\mathcal{F}$ is normal in the extended sense. This finishes the proof.

**The Method II:** We use the $\tan^{-1}$ directly convert the spherical distance to Euclidean distance without using the lemma (but use the Montel’s theorem proved earlier in the class).
"⇒" : As in the proof of Montel’s theorem in the textbook, to prove \( F \) is normal, we only need to consider each compact subset \( K \), i.e. to prove that any sequence \( f_n \subset F \) has a subsequence \( f_{n_k} \) which converges uniformly on \( K \). For each point \( z_0 \in U \), take \( r > 0 \) such that \( \overline{D(z_0, r)} \subset U \). By the Heine-Borel, there are only finite many of such \( D(z_0, r) \) which covers the given compact set \( K \). Hence we only need to consider a neighborhood of some fixed \( z_0 \in U \).

Take \( \delta > 0 \) such that \( \overline{D(z_0, \delta)} \subset U \). From the condition, there is \( M > 0 \) such that

\[ f^\#(z) \leq M, \quad z \in \overline{D(z_0, \delta)}. \]

Consider \( h(r) = \tan^{-1}|f(z_0 + re^{i\theta})|, 0 \leq r \leq \delta \). Then

\[ |h(r) - h(0)| \leq \int_0^r |h'(t)|dt = \int_0^r \frac{\frac{\partial f}{\partial t}|f(z_0 + te^{i\theta})|}{1 + |f(z_0 + te^{i\theta})|^2}dt. \]

Since \( |f| = \sqrt{u^2 + v^2} \) when write \( f = u + iv \),

\[ \left| \frac{\partial}{\partial t}|f(z_0 + te^{i\theta})| \right| = \left| \frac{u \frac{\partial u}{\partial t} + v \frac{\partial v}{\partial t}}{\sqrt{u^2 + v^2}} \right| \leq \sqrt{\left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial v}{\partial t} \right)^2} = \left| \frac{\partial f}{\partial t} \right| = |f'(z_0 + te^{i\theta})|, \]

so

\[ |h(r) - h(0)| \leq \int_0^r \frac{|f'(z_0 + te^{i\theta})|}{1 + |f(z_0 + te^{i\theta})|^2}dt \leq Mr \leq M\delta. \]

By taking \( \delta < \frac{\pi}{12 \pi M} \), we have that, for \( z \in \overline{D(z_0, \delta)} \),

\[ |\tan^{-1}|f(z)| - \tan^{-1}|f(z_0)|| \leq \frac{\pi}{12}. \]

For each \( f \in F \), we devide it into two cases:

**case 1:** \( |f(z_0)| \leq 1 \), then

\[ |\tan^{-1}|f(z)| \leq |\tan^{-1}|f(z_0)| + \frac{\pi}{12} = \frac{\pi}{4} + \frac{\pi}{12} = \frac{\pi}{3}, \]

i.e.

\[ |f(z)| \leq \sqrt{3}. \]

**case 2:** \( |f(z_0)| > 1 \), then

\[ |\tan^{-1}|f(z)| \geq |\tan^{-1}|f(z_0)| - \frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{12} = \frac{\pi}{6}, \]

i.e.

\[ |f(z)| \geq \frac{1}{\sqrt{3}}. \]

7
Now for any sequence \( f_n \subset F \), from the above discussion, there is a subsequence \( f_{n_k} \) such that either \( |f_{n_k}(z)| \leq \sqrt{3} \) for ALL \( f_{n_k} \) or \( |f(z)| \geq \frac{1}{\sqrt{3}} \) ALL \( f_{n_k} \), which in both cases, has a subsequence converges uniformly on \( K \) by Montel’s theorem. This proves the Marty’s theorem.

**Remark about the proof:** You may be concerned about the existence of \( \frac{\partial}{\partial t} |f(z_0 + te^{i\theta})| \) (whether \( |f(z_0 + te^{i\theta})| \) is differentiable). It certainly exists at the point \( t_0 \) with \( f(z_0 + t_0e^{i\theta}) \neq 0 \). Since the holomorphic function \( f \) has finitely many zeros in the compact set \( K \), so the integral \( \frac{f}{1+|f(z_0+te^{i\theta})|^2} dt \) can be taken by removing those points where \( f(z_0+t_0e^{i\theta}) \neq 0 \). Hence our proof goes through. In another way, you can, at first assume that \( f(z_0) \neq 0 \) (otherwise you can take another point which is sufficiently close to \( z_0 \)), and for any fixed \( r \) with \( 0 < r \leq \delta \) take \( \theta \) such that \( f(z_0 + te^{i\theta}) \) has no zeros for \( 0 \leq t \leq r \). Once you prove the bounds for those \( f(z) \), you can prove that it also holds for all \( \theta \) without restriction due to the fact that \( |f(z_0 + te^{i\theta})| \) is continuous in terms of \( \theta \).

**The Method III:** This argument is similar to the the second one, but more straightforward.

“\(\Leftarrow\)” : As in the proof of Montel’s theorem in the textbook, to prove \( F \) is normal, we only need to consider each compact subset \( K \), i.e. to prove that any sequence \( f_n \subset F \) has a subsequence \( f_{n_k} \) which converges uniformly on \( K \). For each point \( z_0 \in U \), take \( r > 0 \) such that \( \overline{D(z_0, r)} \subset U \). By the Heine-Borel, there are only finite many of such \( D(z_0, r) \) which covers the given compact set \( K \). Hence we only need to consider a neighborhood \( D(z_0, \delta) \subset K \). From the assumption, for \( z \in K \),

\[
\frac{|f'(z)|}{1 + |f(z)|^2} \leq C_K,
\]

i.e. \( |f'(z)| \leq C_K(1 + |f(z)|^2) \).

For each \( f \in F \), we devide it into two cases:

**Case 1:** \( |f(z_0)| \leq 1 \). We claim that for \( z \) with \( |z - z_0| \leq \min\{\delta, 1/(5C_K)\} \), we have \( |f(z)| < 2 \). Indeed, if on \( D(z_0, \delta) \), \( |f(z)| < 2 \), then we are done. Otherwise, there is \( z_1 \in D(z_0, \delta) \) with \( |f(z_1)| = 2 \) and on the line segment \( \overline{z_0z_1} \) we have \( |f(z)| < 2 \). Hence

\[
2 = |f(z_1)| \leq |f(z_0)| + \left| \int_{z_0z_1} f'(\zeta)d\zeta \right| \leq 1 + 5C_K|z_1 - z_0|.
\]

Thus \( |z_1 - z_0| > 1/(5C_K) \). This proves the claim.
Case 2: $|f(z_0)| \geq 1$. In this case, we claim that $z$ with $|z - z_0| \leq \min\{\delta, 1/(5C_K)\}$, we have $|f(z)| > 1/2$. Indeed, if on $D(z_0, \delta)$, $|f(z)| > 1/2$, then we are done. Otherwise, there is $z_1 \in D(z_0, \delta)$ with $|f(z_1)| = 1/2$ and on the line segment $\overline{z_0 z_1}$ we have $|f(z)| > 1/2$. Using the fact that

$$\frac{|\left(\frac{1}{f(z)}\right)'|}{1 + \left|\frac{1}{f(z)}\right|^2} \leq \frac{|f'(z)|}{1 + |f(z)|^2} \leq C_K$$

we have that, for $z \in D(z_0, \delta)$

$$|\left(\frac{1}{f(z)}\right)'| \leq 5C_K.$$  

Hence

$$2 = \frac{1}{|f(z_1)|} \leq \frac{1}{|f(z_0)|} + \left|\int_{z_0 z_1} \left(\frac{1}{f(\zeta)}\right)' \, d\zeta\right| \leq 1 + 5C_K |z_1 - z_0|.$$  

Thus $|z_1 - z_0| > 1/(5C_K)$. This proves the claim.

Now for any sequence $f_n \subset F$, from the above discussion, there is a subsequence $f_{nk}$ such that either $|f_{nk}(z)| < 2$ for ALL $f_{nk}$ or $|f(z)| \geq 1/2$ ALL $f_{nk}$, which in both cases, has a subsequence converges uniformly on $K$ by Montel’s theorem. This proves the Marty’s theorem.

(Zalcman’s theorem)

Let $F$ be a family of holomorphic functions on the unit disc $D$ of $\mathbb{C}$. Assume $F$ is not normal in the extended sense, then there exist an $r$ with $0 < r < 1$, a sequence of points $\{z_n\}$ with $|z_n| < r$, a sequence $\{f_n\} \subset F$, and a sequence of positive numbers $\{\rho_n\}$ with $\lim_{n \to +\infty} \rho_n = 0$ such that $f_n(z_n + \rho_n \xi) \to g(\xi)$ uniformly on every compact subset $K$ in $\mathbb{C}$, and $g$ is not constant.

Proof.

By the previous Marty’s theorem, since $F$ is not normal, then there should be a compact subset $K$ of $D$, such that Marty’s criterion is violated on it; moreover, we can take $K$ as one of the exhaustion $\{K_n\}$, where $K_n = \overline{D(0, 1 - 1/n)}$. So, equivalently, we actually have an $r_0$, with $0 < r_0 < 1$, a sequence of points $\{z_n^*\}$, with $z_n^* < r_0$, and a sequence of holomorphic functions $\{f_n\} \subset F$, such that

$$f_n^*(z_n^*) \to +\infty, \quad \text{as } n \to +\infty.$$  

Next, we set, for any fixed $r_0 < r < 1$,

$$M_n = \max_{|z| \leq r} \left(1 - \frac{|z|^2}{r^2}\right) f_n^*(z).$$  

9
Since \((1 - \frac{|z|^2}{r^2}) f_n^\#(z)\) is continuous for every \(n\), and the set \(\{ z : |z| \leq r \}\) is compact, then the above maximum exists for every \(n\), i.e. there is a sequence \(\{z_n\}\) such that
\[
M_n = \left(1 - \frac{|z_n|^2}{r^2}\right) f_n^\#(z_n)
\]
In addition, because
\[
\left(1 - \frac{|z_n^*|^2}{r^2}\right) f_n^\#(z_n^*) \geq \left(1 - \frac{r_0^2}{r^2}\right) f_n^\#(z_n^*) \rightarrow +\infty
\]
So, indeed, we obtain that \(M_n \rightarrow +\infty\).

Now, set
\[
\delta_n = \frac{1}{M_n} \left(1 - \frac{|z_n|^2}{r^2}\right) = \frac{1}{f_n^\#(z_n)}
\]
So,
\[
\frac{\delta_n}{r - |z_n|} = \frac{r + |z_n|}{r^2 M_n} \leq \frac{2r}{r^2 M_n} \rightarrow 0, \quad \text{as} \ n \rightarrow +\infty
\]
Also, \(\delta_n \rightarrow 0\) as \(n \rightarrow +\infty\).

Now, we shall construct the following sequence of holomorphic functions,
\[
g_n(\xi) := f_n(z_n + \delta_n \xi)
\]
and it is easily seen that \(g_n\) is defined on \(|\xi| < R_n\), where \(R_n = (r - |z_n|)/\delta_n \rightarrow +\infty\).

To see that the limit function \(g(\xi) = \lim_{n \rightarrow +\infty} g_n(\xi)\) is an entire function, we will apply the Marty’s theorem as follows,

First, compute \(g^\#\) for each \(g_n\), we have that
\[
g_n^\#(\xi) = \frac{\delta_n |f_n'(z_n + \delta_n \xi)|}{\sqrt{1 + |f_n(z_n + \delta_n \xi)|^2}} = \frac{f_n^\#(z_n + \delta_n \xi)}{f^\#(z_n)}
\]

Second, on \(|\xi| \leq R < R_n\), since \(|z_n + \delta_n \xi| \leq r^* < r\) for each \(n\), we can evaluate \(g_n^\#(\xi)\) as
\[
g_n^\#(\xi) \leq \frac{1 - \frac{|z_n|}{r^*}}{1 - \frac{|z_n + \delta_n \xi|^2}{r^*}} \leq \frac{r^2}{r^2 - (r^*)^2}
\]
Hence, by the Marty’s theorem, we know that \(\{g_n\}\) is normal, which implies that there is a subsequence, \(\{g_{n_k}\}\), which converges uniformly to \(g(\xi)\) on every compact subset of \(C\), so \(g\) is entire.
Moreover, it is easily calculated that $g_n^\#(0) = 1$ for each $n$, so $g^\#(0) = 1$, implying that $g'(0) \neq 0$. Thus, $g$ is nonconstant.

This completes the proof of the Zalcman’s theorem.

\[\square\]

(Generalized Montel’s Theorem)

Let $\mathcal{F}$ be a family of holomorphic functions on a region $U$ of $\mathbb{C}$. Assume that $f \neq 0, 1$ for each $f \in \mathcal{F}$, then $\mathcal{F}$ is a normal family.

Proof.

If $U = \mathbb{C}$, then the proof becomes trivial, since by the little Picard’s theorem, each function in $\mathcal{F}$ is constant functions, so $\mathcal{F}$ is normal.

If $U \subsetneq \mathbb{C}$, then by the Riemann Mapping Theorem, we can use a conformal mapping $G$ to map $U$ to the unit disc $D$, and prove the new family, $\{G \circ f : f \in \mathcal{F}\}$ is normal, so $\mathcal{F}$ is also normal.

Hence, without loss of generality, we can suppose that $U = D$, then the proof goes as the following,

Suppose $\mathcal{F}$ is not normal, then by the Zalcman’s theorem, there exist an $r$ with $0 < r < 1$, a sequence of points $\{z_n\}$ with $|z_n| < r$, a sequence $\{f_n\} \subset \mathcal{F}$, and a sequence of positive numbers $\{\rho_n\}$ with $\lim_{n \to +\infty} \rho_n = 0$ such that $f_n(z_n + \rho_n \xi) \to g(\xi)$ uniformly on every compact subset $K$ in $\mathbb{C}$, and $g$ is not constant entire function.

However, the fact that each $f_n$ omits $0, 1$ tells us that $g$ omits $0, 1$. So, by the little Picard’s theorem, we conclude that $g$ is constant function, but this contradicts the result of Zalcman’s theorem, which states that $g$ is nonconstant.

Finally, due to the contradiction, we should have $\mathcal{F}$ is a normal family.

\[\square\]