16. Prove: If $u$ is a complex-valued harmonic function, then the real and the imaginary parts of $u$ are harmonic. Conclude that a complex-valued function $u$ is harmonic if and only if $ar{u}$ is harmonic.

Proof. First, we set $u(x, y) = f(x, y) + ig(x, y)$, where $f$ and $g$ are both real-valued, then by the condition that $u$ is a harmonic, it is easily seen that both $f$ and $g$ are in $C^2$. Also, taking the Laplacian of $u$ and we obtain

$$\Delta u = \Delta f + i\Delta g = 0$$

which clearly implies that $\Delta f = \Delta g = 0$, so $f$ and $g$ are both harmonic.

Next, if $u$ is harmonic, then $f$ and $g$ are both harmonic, and it is easily seen that $\bar{u} = f - ig$, their linear combination, is in $C^2$. Moreover, since $\Delta \bar{u} = \Delta f - i\Delta g$, then $\Delta f = \Delta g = 0$ implies that $\Delta u = 0$. So $\bar{u}$ is also harmonic.

Conversely, if $\bar{u}$ is harmonic, then $f$ and $g$ are also harmonic, which directly means that $u$ is harmonic.

19. Prove that there is no nonconstant harmonic functions $u : \mathbb{C} \to \mathbb{R}$ such that $u(z) \leq 0$ for all $z \in \mathbb{C}$.

Proof. Since $\mathbb{C}$ is simply connected, we know that there is a harmonic function $v : \mathbb{C} \to \mathbb{R}$, such that $f = u + iv$ is holomorphic on $\mathbb{C}$.

Thus, consider the function $e^f$, which is also holomorphic on $\mathbb{C}$, we have

$$|e^f| = |e^u + iv| = e^u \leq 1$$

The last inequality holds, because $u \leq 0$ on $\mathbb{C}$.
So, by the Liouville’s theorem, \( e^f \) is constant on \( \mathbb{C} \), which implies that \( u \) is constant. This means that there is no nonconstant function that satisfies the given condition in the problem.

**Remark:** You can also use the Harnack inequality to prove it.

25. Compute a formula analogous to the Poisson integral formula, for the region \( U = \{ z : \Im z > 0 \} \) (the upper half plane), by mapping \( U \) conformally to the unit disc.

The formula is as follows: Suppose \( u \) is harmonic on \( U = \{ z : \Im z > 0 \} \), continuous and bounded on \( \overline{U} = \{ z : \Im z \geq 0 \} \), then, for \( z = x + yi \) with \( y > 0 \), we have

\[
u(z) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{y}{(x-t)^2 + y^2} u(t) dt\]

*Proof.* First, we set \( \varphi(z) = \frac{z+i}{z-i} \), the Cayley transform, then \( \varphi \) maps \( U \) conformally onto the unit disc, and the boundary \( \mathbb{R} \) to \( \{ z : |z| = 1 \} \setminus \{ z = 1 \} \).

Second, consider the mapping \( u \circ \varphi^{-1} \), which is holomorphic on \( D(0,1) \), and continuous on \( \overline{D}(0,1) \) since \( u \) is bounded, then we have the Poisson integral formula on the unit disc: if \( a \in D(0,1) \), then

\[
u \circ \varphi^{-1}(a) = \frac{1}{2\pi} \oint_{\{ |\zeta| = 1 \} \setminus \{ \zeta = 1 \}} \frac{1-|a|^2}{|\zeta-a|^2} u \circ \varphi^{-1}(\zeta) \frac{d\zeta}{i\zeta} \quad (*)\]

Next, make substitution of variables by setting \( \zeta = \varphi(t) \), where \( t \in \mathbb{R} \), and \( a = \varphi(z) \), we obtain

\[
u \circ \varphi^{-1}(a) = u(z) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1-|\varphi(z)|^2}{|\varphi(t) - \varphi(z)|^2} u(t) \frac{\varphi'(t) dt}{i\varphi(t)}
= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1-\left|\frac{z-i}{t+i}-\frac{z-i}{z+i}\right|^2}{\left|\frac{t-i}{t+i}-\frac{z-i}{z+i}\right|^2} u(t) \frac{2dt}{t^2 + 1}
= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{[(y+1)^2 - (y-1)^2](t^2 + 1)}{2((t-x)^2 + y^2)} \cdot \frac{2}{t^2 + 1} dt
= \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{y}{(t-x)^2 + y^2} u(t) dt\]

This proves the desired formula.

*Remark.* Here, for the desired formula we have proved, since it involves both \( +\infty \) and \( -\infty \) as the upper and lower limit, we actually need to use a limit process to define it (e.g. C.P.V is a nice way to define, as \( u \) is bounded). Also, as \( \infty \) is
not included in the upper half plane, the image of it under Cayley’s transform omits the boundary point \( z = 1 \), so we have to integrate on \( \{ |\zeta| = 1 \} \setminus \{ \zeta = 1 \} \) in equality (*), which is defined via a limit process corresponding to our final formula. But this (formula involving the limit) seems a little different from averaging formula, which is purely equality. However, the following lemma will fix this difference.

**Lemma.** Suppose that \( u(z) \) is harmonic on \( D(0, 1) \), and continuous on \( \overline{D}(0, 1) \) except at \( z = 1 \), then for \( |z| < 1 \), we have that

\[
u(z) = \frac{1}{2\pi} \int_{\{ \zeta = 1 \} \setminus \{ \zeta = 1 \}} \frac{1 - |z|^2}{|\zeta - z|^2} u(\zeta) \frac{d\zeta}{i\zeta} = \frac{1}{2\pi} \lim_{\epsilon \to 0} \int_{\epsilon}^{2\pi - \epsilon} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} u(e^{i\theta}) d\theta
\]

**Proof.** Fix a \( z \in D(0, 1) \), then since \( u \) is harmonic on \( D(0, 1) \), then for all \( 0 < r < 1 \), we have

\[
u(z) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} u(e^{i\theta}) d\theta
\]

Then, consider

\[
|\nu(z) - \frac{1}{2\pi} \int_{r}^{2\pi - r} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} u(e^{i\theta}) d\theta| = \frac{1}{2}\left| \int_{r}^{2\pi - r} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} [u(re^{i\theta}) - u(e^{i\theta})] d\theta \right|
\]

\[
+ \int_{0}^{r} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} u(e^{i\theta}) d\theta + \int_{2\pi - r}^{r} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} u(e^{i\theta}) d\theta \right|
\]

Next, because \( u \) is uniformly continuous on \( \{ z = re^{i\theta} : r \leq 1, \epsilon \leq \theta \leq 2\pi - \epsilon \} \), and \( u \) us bounded on \( \overline{D}(0, 1) \setminus \{1\} \), by letting \( r \to 1, \epsilon \to 0 \), we can make the right hand side of the above equality arbitrarily small, that is

\[
u(z) = \frac{1}{2\pi} \lim_{\epsilon \to 0} \int_{\epsilon}^{2\pi - \epsilon} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} u(e^{i\theta}) d\theta
\]

The lemma is proved.

\[\Box\]

27. Let \( P(z, \zeta) \) be the Poisson kernel for the disc. If you write \( z = re^{i\theta} \) and \( \zeta = e^{i\phi} \), then you can relate the formula for the Poisson kernel that was given in the text to the new formula

\[
P(z, \zeta) = \frac{1}{2\pi} \frac{1 - |z|^2}{|z - \zeta|^2} = \frac{1}{2\pi} \frac{|\zeta|^2 - |z|^2}{|z - \zeta|^2}
\]
Do so.

Now calculate \( \Delta_z P(z, \zeta) \) for \( z \in D(0, 1) \) to see that \( P \) is harmonic in \( z \).

**Proof.** From the expression of the Poisson formula, it is obvious that

\[
P(z, \zeta) = \frac{1}{2\pi} \frac{1 - |z|^2}{|z - \zeta|^2} = \frac{1}{2\pi} |\zeta|^2 - |z|^2
\]

So what left to prove is that \( P(z, \zeta) \) is harmonic w.r.t \( z \).

Since \( \Delta_z = 4\frac{\partial^2}{\partial z \partial \bar{z}} \), we have

\[
\Delta_z P(z, \zeta) = 4\frac{\partial^2}{\partial z \partial \bar{z}} \left( \frac{1}{2\pi} |\zeta|^2 - |z|^2 \right)
= \frac{2}{\pi} \frac{\partial^2}{\partial z \partial \bar{z}} \left( \frac{\zeta \bar{\zeta} - z \bar{z}}{(z - \zeta)(\zeta - \bar{\zeta})} \right)
= \frac{2}{\pi} \frac{\partial^2}{\partial z \partial \bar{z}} \left( \frac{\zeta}{\zeta - z} + \frac{\bar{\zeta}}{\bar{\zeta} - z} - 1 \right)
= \frac{2}{\pi} \frac{\partial^2}{\partial z \partial \bar{z}} \left( \frac{\zeta}{\zeta - z} \right) + \frac{2}{\pi} \frac{\partial^2}{\partial z \partial \bar{z}} \left( \frac{\bar{\zeta}}{\bar{\zeta} - z} \right) - \frac{2}{\pi} \frac{\partial^2}{\partial z \partial \bar{z}}(1)
= 0
\]

This shows that \( P(z, \zeta) \) is harmonic w.r.t \( z \).

\( \square \)

28. If \( h_1, h_2, \ldots \) are harmonic on \( U \subseteq \mathbb{C} \) and if \( \{b_j\} \) converges uniformly on compact subsets of \( U \), then prove that the limit function \( h_0 \) is harmonic.

**Proof.** For any \( P \in U \), and any \( \delta > 0 \) such that \( \bar{D}(P, \delta) \subseteq U \), we have, by the fact that \( \{h_n\} \) is harmonic, for each \( z \in D(P, \delta) \),

\[
h_n(z) = \int_0^{2\pi} \frac{\delta^2 - |z - P|^2}{|z - P - \delta e^{i\theta}|^2} h_n(P + \delta e^{i\theta}) d\delta
\]

Then, as \( h_n \to h_0 \) uniformly on \( \bar{D}(P, \delta) \), we can choose \( n \) large enough such that \( |h_n(P + \delta e^{i\theta}) - h_0(P + \delta e^{i\theta})| < \epsilon \), for all \( \theta \in [0, 2\pi] \), so we have

\[
\left| h_n(z) - \frac{1}{2\pi} \int_0^{2\pi} \frac{\delta^2 - |z - P|^2}{|z - P - \delta e^{i\theta}|^2} h_0(P + \delta e^{i\theta}) d\delta \right|
\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{\delta^2 - |z - P|^2}{|z - P - \delta e^{i\theta}|^2} \left| h_n(P + \delta e^{i\theta}) - h_0(P + \delta e^{i\theta}) \right| d\delta
\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{\delta^2 - |z - P|^2}{|z - P - \delta e^{i\theta}|^2} \left| h_n(P + \delta e^{i\theta}) - h_0(P + \delta e^{i\theta}) \right| d\delta
\leq \epsilon \cdot \frac{1}{2\pi} \int_0^{2\pi} \frac{\delta^2 - |z - P|^2}{|z - P - \delta e^{i\theta}|^2} d\delta
= \frac{\epsilon}{2\pi}
\]

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Here, we also used the fact that \( \int_0^{2\pi} \frac{\delta^2 - |z - P|^2}{|z - P|^2} d\delta = 1. \)

Thus, we know that \( h_n(z) \to h_0(z) \); therefore, by the uniqueness of limit, this implies that

\[
h_0(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\delta^2 - |z - P|^2}{|z - P|^2} h_0(P + \delta e^{i\theta}) d\delta
\]

Finally, since our \( z \) and \( \delta \) is arbitrarily chosen, we conclude that \( h_0 \) is harmonic on \( U \).

41. Prove that if \( f \) is \( C^2 \) on an open set \( U \) and \( f \) is subharmonic, then \( \Delta f \geq 0 \) on \( U \).

**Proof.** First, since \( f \) is \( C^2 \), for \( a \in U \), and a small neighborhood around it, say \( D(a,R) \), we can expand \( f \) as second order Taylor polynomial in complex variables \( z \) and \( \bar{z} \):

\[
f(z) = f(a) + \frac{\partial f}{\partial z}(a)(z - a) + \frac{\partial f}{\partial \bar{z}}(a)(\bar{z} - \bar{a}) + \frac{1}{2} \frac{\partial^2 f}{\partial z \partial \bar{z}}(a)(z - a)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial \bar{z}^2}(a)(\bar{z} - \bar{a})^2 + O(|z - a|^3)
\]

Here, \( h = O(|z - a|^3) \) means \( |h| \leq C|z - a|^3 \) on \( D(a,R) \) for some \( C > 0 \).

Thus, set \( z = a + re^{i\theta} \) and integrate the both sides of the above equation, we obtain

\[
f(a) \leq \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta = f(a) + \frac{\partial^2 f}{\partial z \partial \bar{z}}(a)r^2 + O(r^3)
\]

By letting \( r \to 0 \), we get \( \Delta f \geq 0 \), and this completes the proof.

43. **(Maximum Principle for Subharmonic functions)**

If \( f \) is subharmonic on \( U \), and if there is a \( P \in U \) such that \( f(P) \geq f(z) \) for any \( z \in U \), then \( f \) is constant.

**Proof.** First, we set \( s = f(P) \), and consider the set \( M = \{ z \in M : f(z) = s \} \).

We shall prove that \( M \) is both open and closed. Clearly, \( M \) is non-empty.
Since $f$ is continuous, $M$ is closed. On the other hand, for any $w \in M$ and $D(w, r) \subseteq U$, we have the following inequalities, by the fact that $f$ is subharmonic:

$$s = f(w) \leq \frac{1}{2\pi} \int_0^{2\pi} f(w + re^{i\theta}) d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} sd\theta = s$$

So we actually obtain that $\frac{1}{2\pi} \int_0^{2\pi} f(w + re^{i\theta}) d\theta = s$, and since for any $\theta \in [0, 2\pi]$, $f(w + re^{i\theta}) \leq s$, we know that $f(w + re^{i\theta}) = s$ for all $\theta \in [0, 2\pi]$. This means that $M$ is open.

In sum, $M$ is non-empty, open, and closed, we conclude that $M = U$, and therefore $f$ is constant on $U$.

44. If $u$ is harmonic on $U \subseteq \mathbb{C}$, then $u$ satisfies a minimum principle as well as a maximum principle. However, subharmonic functions do not satisfy a minimum principle. Illustrate this claim.

Proof. To explain this claim, we shall construct a counter-example.

Set $r > 0$, and consider the following function on the closed unit disc $\overline{D}(0, 1)$:

$$f(z) = \log(r + |z|^2)$$

Then, it is easy to check that

$$\Delta f(z) = 4 \frac{\partial^2}{\partial z \partial \overline{z}} (\log(r + |z|^2)) = \frac{r}{(r + |z|^2)^2} > 0$$

Thus, $f$ is subharmonic; however, it violates the minimum principle, since $\min f$ is attained at the origin, an interior point.

Remark: Another simple example is $u(z) = |z|^2$.

47. Let $U \subseteq \mathbb{C}$ be a connected open set. Let $f : U \to \mathbb{R}$ be subharmonic. Suppose further that $V$ is open and $F : V \to U$ is holomorphic. Prove that $f \circ F$ is subharmonic. What happens if $F$ is only harmonic?

Proof. For any $P \in V$ and $\overline{D}(P, r) \subseteq V$, and any harmonic function $h$ on $\overline{D}(P, r)$, we need to prove that if $f \circ F - h \leq 0$ on $\partial D(P, r)$, then $f \circ F - h \leq 0$ on $D(P, r)$.
To show this, we first assume that $F$ is one-to-one, thus it is biholomorphic from $V$ to $\text{ran}(F)$. We need to show that for any $D(P, r) \subseteq V$, and any harmonic function $h$ on $D(P, r) \subseteq V$, such that $h \geq f \circ F$ on $\partial D(0, 1)$, then $h \geq f \circ F$ on $D(0, 1)$.

Consider $f \circ F - h$, we can rewrite it as

$$f \circ F(z) - h \circ F^{-1}(F(z))$$

as $F$ is biholomorphic.

Then, notice that $F(\partial D(P, r))$ is also a Jordan curve in $U$, which is the boundary of $F(V)$, hence $f(F(z)) - h \circ F^{-1}(F(z)) \leq 0$ means that $f - h \circ F^{-1} \leq 0$ on $F(\partial D(P, r))$, and by the fact that $h$ is subharmonic, we know that $f - h \circ F^{-1} \leq 0$ on $F(V)$. This is exactly what we desire, i.e., $f \circ F(z) - h(z) \leq 0$ in $D(P, r)$, so we proved that $f \circ F$ is also subharmonic.

Now, suppose that $F$ is not one-to-one, if $F$ is constant, then proof is trivial. If $F$ is nonconstant, then there exists a point $P$ such that $F'(P) = 0$. In this case, we know that there is an injective holomorphic function $g$ on $D(P, r)$ such that $F = g^k + F(P)$, so what we need to prove is that $f(z^k + F(P))$ is subharmonic at $z = 0$. This can be seen from the following:

$$\frac{1}{2\pi} \int_0^{2\pi} f(\epsilon e^{i\theta} + F(P))d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(\epsilon e^{i\zeta} + F(P)) \frac{d\zeta}{k}$$
$$= \frac{1}{2\pi} \int_0^{2\pi} f(\epsilon e^{i\zeta} + F(P))d\zeta$$
$$\leq f(F(P))$$

This means $f(z^k + F(P))$ is subharmonic at $z = 0$, i.e., $f \circ F$ is subharmonic at $P$.

For a point $Q$ such that $F'(Q)$ is not zero, we know that there is a neighborhood around $Q$ such that $F$ is injective on it, and the proof is the same as the above case where $F$ is one-to-one.

Thus, in sum, we conclude that $f \circ F$ is also subharmonic, given $F$ is holomorphic.

However, if $F$ is only harmonic, then the claim may not be true. To see this, consider the function $f(z) = 2x^2 - y^2$. We know that it is subharmonic on $\mathbb{C}$. Since it is a composition of a subharmonic function and an increasing convex function.

Next, take $F(x, y) = x + xyi$, then we have that $f \circ F(x, y) = 2x^2 - x^2y^2$, which is not subharmonic, since $\Delta f \circ F < 0$ at some points, and this explains why harmonic functions cannot transform subharmonic functions into subharmonic functions.
69. Let \( f : U \to \mathbb{R} \) be a \( C^2 \) function on an open set in \( U \subseteq \mathbb{C} \).

(a). Recall that if \( \Delta f > 0 \) at a point \( P \), then \( f \) cannot have a local maximum at \( P \). Use this observation to deduce that if \( \Delta f > 0 \) everywhere on \( U \), then \( f \) is subharmonic.

(b). If \( \Delta f \geq 0 \) everywhere, then, for each \( \epsilon > 0 \), \( \Delta (f + \epsilon |z|^2) > 0 \) everywhere. Use a limiting argument and part (a) to deduce that if \( \Delta f \geq 0 \) everywhere, then \( f \) is subharmonic.

**Proof.** (a). Given that \( \Delta f > 0 \) everywhere on \( U \), then for any neighborhood \( \bar{D}(P, r) \subseteq U \), and any harmonic function \( h \) on \( \bar{D}(P, r) \), such that \( u - h \leq 0 \) on \( \partial D(P, r) \), we need to show that \( u - h \leq 0 \) on \( D(P, r) \).

We suppose that there is a \( Q \in D(P, r) \) that violates our desired result, that is, \( u(Q) - h(Q) > 0 \); however, this implies that \( u - h \) attains its maximum inside \( D(P, r) \), and without loss of generality, we can assume that \( u - h \) attains its maximum at \( Q \).

Next, from the knowledge in mathematical analysis, we know that if \( u - h \) attains maximum at \( Q \), then its Hessian matrix at \( Q \) cannot be positive definite (indeed it can only be negative semi-definite) at \( Q \). So we have that

\[
\frac{\partial^2 (u - h)}{\partial x^2}(Q) \leq 0, \quad \frac{\partial^2 (u - h)}{\partial y^2}(Q) \leq 0 \quad (\ast)
\]

But we already have the given condition that \( \Delta u > 0 \), from which we know that \( \Delta (u - h) > 0 \). This contradicts the above result \((\ast)\) derived by our assumption. Hence, due to this contradiction, \( u - h \) cannot attain its maximum inside \( D(P, r) \), and this implies that \( u - h \leq 0 \) inside \( D(P, r) \). So, by definition, we conclude that \( u \) is subharmonic on \( U \).

(b). Now we only have that \( \Delta f \geq 0 \), so we consider the function \( f + \epsilon |z|^2 \). It is easy to see that \( \Delta (f + \epsilon |z|^2) > 0 \) for every \( \epsilon > 0 \), and therefore \( f + \epsilon |z|^2 \) is subharmonic on \( U \).

Next, we notice that for each \( \bar{D}(P, r) \subseteq U \), \( f + \epsilon |z|^2 \) converges uniformly to \( f \) on it. Moreover, for each \( \epsilon > 0 \), we have that

\[
f(P) + \epsilon |P|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} f(P + re^{i\theta}) + \epsilon |P + re^{i\theta}|^2 d\theta \quad (\ast)
\]

Then, by the uniform convergence,

\[
f(P) + \epsilon |P|^2 \to f(P), \quad \text{as} \ \epsilon \to 0
\]

\[
\frac{1}{2\pi} \int_0^{2\pi} f(P + re^{i\theta}) + \epsilon |P + re^{i\theta}|^2 d\theta \to \frac{1}{2\pi} \int_0^{2\pi} f(P + re^{i\theta}) d\theta, \quad \text{as} \ \epsilon \to 0
\]

This directly implies \( f(P) \leq \frac{1}{2\pi} \int_0^{2\pi} f(P + re^{i\theta}) d\theta \), by the inequality \((\ast)\). So we know that \( f \) is subharmonic.