Part A:

(1). (Residue Formula)

Suppose $f$ is a holomorphic function on a holomorphically simply connected domain $U \subseteq \mathbb{C}$, except for singularities $z_1, z_2, \cdots, z_n$, and $\gamma$ is a piecewise closed $C^1$ curve in $U$, but not passing through any $z_k$. Then the following equality holds:

$$\oint_\gamma f(\zeta) d\zeta = 2\pi i \sum_{k=1}^{n} \text{Res}_f(z_k) \cdot n(\gamma, z_k)$$

where $n(\gamma, z_k)$ denotes the winding number of $\gamma$ about each $z_k$.

Proof. $U \subseteq \mathbb{C}$ is said to be **holomorphically simply connected** if every holomorphic function on $U$ has an anti-derivative (or primitive). Hence, $\oint_\gamma f dz = 0$ if $f$ is holomorphic on $U$ (because let $F$ with $F' = f$, then $\oint_\gamma f dz = F(\gamma(b)) - F(\gamma(a)) = 0$ since $\gamma$ is closed. We’ll use this fact to prove the residue theorem.

First, for each $z_k$, we set the principal part of the Laurent expansion of it to be $S_k$, then consider the following auxiliary function:

$$F(z) = f - (S_1 + \cdots + S_n), \quad G(z) = S_1 + \cdots + S_n$$

Notice that $G = (S_1 + \cdots + S_n)$ is defined on $U \setminus \{z_1, \cdots, z_n\}$, which is the same as $f$. Moreover, for each singularity $z_k$, if we consider the Laurent expansion around it, the $f - S_k$ has just the regular part left, hence, by the Riemann’s theorem, $z_k$ is removable singularity for $F = f - (S_1 + \cdots + S_n)$; therefore, we can regard $F$ as a holomorphic function on $U$.

Next, by the simple connectedness of $U$, and the Cauchy’s theorem, we have that $\oint_\gamma F(z) dz = 0$.

Now, consider $\oint_\gamma G(z) dz = \sum_{k=1}^{n} \oint_\gamma S_k dz$, we know that each $S_k$ is the principal part of the Laurent expansion of $f$ at $z_k$. So $S_k(z) = \sum_{l=1}^{\infty} a^k_{l}(z - z_k)^{-l}$, and by
the fact that $\gamma$ is compact and does not pass through $z_k$, we know $\sum_{l=1}^{\infty} a_{-l}^k (z-z_k)^{-l}$ converges uniformly on $\gamma$, thus, using the fact that $\oint_{\gamma} (z-z_k)^{-l} = 0$ for $k \neq 1$,

$$\oint_{\gamma} S_k(z)dz = \sum_{l=1}^{\infty} \oint_{\gamma} a_{-l}^k (z-z_k)^{-l}dz = \oint_{\gamma} a_{-1}^k (z-z_k)^{-1}dz = 2\pi i \text{Res}_f(z_k) \cdot n(\gamma, z_k)$$

Therefore, in sum, combine the above results, we conclude that

$$\oint_{\gamma} f(z)dz = \oint_{\gamma} (F(z) + G(z))dz = \oint_{\gamma} G(z)dz = \sum_{k=1}^{n} \oint_{\gamma} S_k dz = 2\pi i \sum_{k=1}^{n} \text{Res}_f(z_k) \cdot n(\gamma, z_k).$$

This proved our desired Residue Formula.

$\square$

Let $g = f(\zeta)/(\zeta - z_0)$. Observe that $\text{Res}_g(z_0) = f(z_0)$ by the Laurent expansion of $g = f(\zeta)/(\zeta - z_0)$ at the point $z_0$. Apply the above residue theorem to $g$ to yield that, for any (piecewise smooth) closed simple curve, $\gamma$ with $z_0 \in \text{int}(\gamma)$, since $n(\gamma, z_0) = 1$,

$$f(z_0) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)}d\zeta.$$

Thus the above Cauchy’s integral formula is proved.

$\square$

To prove the argument principle, we first suppose $f$ is a meromorphic function on a open set $U$, $D(P, r) \subseteq U$, and $f$ has no poles or zeros on $\partial D(P, r)$, then we shall show that

$$\frac{1}{2\pi i} \oint_{\partial D(P, r)} \frac{f'(z)}{f(z)}dz = \sum_{j=1}^{p} n_j - \sum_{k=1}^{q} m_k$$

where $n_1, \ldots, n_p$ are the multiplicities of zeros $z_1, \ldots, z_p$ of $f$ in $D(P, r)$, and $m_1, \ldots, m_p$ are the poles $w_1, \ldots, w_q$ of $f$ in $D(P, r)$.

To see this, for the function $h(z) = f'(z)/f(z)$, we know that the possible singularities (actually poles) for it are $z_1, \ldots, z_p$ and $w_1, \ldots, w_q$. Next, for any
zero of $f$, $z_k$, with multiplicity $n_k$, on a neighborhood of which we can write $f$ as $f(z) = (z - z_k)^{n_k}g(z)$, where $g$ is holomorphic and has no zeros on the neighborhood. Then

$$f'(z) f(z) = \frac{n_k(z - z_k)^{n_k-1}g(z) + (z - z_k)^{n_k}g'(z)}{(z - z_k)^{n_k}g(z)}$$

So it is easily seen that $\text{Res}_g(z_k) = n_k$.

Similarly, for any pole of $f$, $w_k$, with multiplicity $m_k$, on a neighborhood of which we can write $f$ as $f(z) = g(z)/[(z - w_k)m_k]$, where $g$ is holomorphic and has no zeros on the neighborhood. Then

$$f'(z)/f(z) = -\frac{m_k(z - w_k)^{m_k-1}g(z) + (z - w_k)^{m_k}g'(z)}{(z - w_k)^{m_k}g(z)}$$

So it is easily seen that $\text{Res}_g(w_k) = -m_k$.

Thus, applying the residue formula, we have that

$$\oint_{\partial D(P,r)} \frac{f'(z)}{f(z)} dz = 2\pi i \left( \sum_{j=1}^{p} n_j - \sum_{k=1}^{q} m_k \right)$$

which is exactly our desired agreement principle formula.

(2). (Laurent Expansion for Holomorphic functions on annulus)

Suppose that $1 \leq r_1 < r_2 \leq \infty$, and $f$ is holomorphic on $D(P, r_2) \setminus \bar{D}(P, r_1)$, then the following series

$$\sum_{k=-\infty}^{+\infty} a_k(z - P)^k$$

converges to $f$ on $D(P, r_2) \setminus \bar{D}(P, r_1)$, where

$$a_j = \frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f(\zeta)}{(\zeta - P)^{j\!+\!1}} d\zeta, \quad \text{for any } r_1 < r_2$$

Moreover, for any pair of number $s_1$, $s_2$, such that $r_1 < s_1 < s_2 < r_2$, the above series converges absolutely and uniformly on $D(P, s_2) \setminus \bar{D}(P, s_1)$

Proof.

First, apply the Cauchy’s integral formula on $\bar{D}(P, s_2) \setminus D(P, s_1)$, and we have that for any $z \in D(P, s_2) \setminus \bar{D}(P, s_1)$

$$f(z) = \frac{1}{2\pi i} \oint_{|\zeta - P|=s_1} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \oint_{|\zeta - P|=s_2} \frac{f(\zeta)}{\zeta - z} d\zeta$$

3
For the first integral, since $|z - P| < |\zeta - P|$ on $|\zeta - P| = s_2$, we have the following:

$$\int_{|\zeta - P| = s_2} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{|\zeta - P| = s_2} \frac{1}{\zeta - P} \cdot \frac{f(\zeta)}{1 - \frac{z - P}{\zeta - P}} d\zeta$$

$$= \int_{|\zeta - P| = s_2} \frac{f(\zeta)}{\zeta - P} \sum_{k=0}^{+\infty} \left( \frac{z - P}{\zeta - P} \right)^k d\zeta$$

$$= \int_{|\zeta - P| = s_2} \sum_{k=0}^{+\infty} f(\zeta)(z - P)^k \left( \frac{\zeta - P}{\zeta - P} \right)^{k+1} d\zeta$$

$$= \sum_{k=0}^{+\infty} \left( \int_{|\zeta - P| = s_2} \frac{f(\zeta)}{(\zeta - P)^{k+1}} d\zeta \right) (z - P)^k$$

Here, the sum and the integral can commute, because the series $\sum_{k=0}^{+\infty} \left( \frac{z - P}{\zeta - P} \right)^k$ converges uniformly to $1/(1 - (z - P)/(\zeta - P))$ on the compact set $|\zeta - P| = s_2$.

Similarly, on $|\zeta - P| = s_1$, notice that $|z - P| > s_1$, we have

$$\frac{1}{2\pi i} \int_{|\zeta - P| = s_1} \frac{f(\zeta)}{\zeta - z} d\zeta = -\int_{|\zeta - P| = s_1} \frac{1}{z - P} \cdot \frac{f(\zeta)}{1 - \frac{z - P}{\zeta - P}} d\zeta$$

$$= -\sum_{k=0}^{+\infty} \left( \int_{|\zeta - P| = s_1} f(\zeta)(z - P)^k d\zeta \right) (z - P)^{-k-1}$$

$$= -\sum_{k=-\infty}^{k=-1} \left( \int_{|\zeta - P| = s_1} f(\zeta) \frac{(z - P)^k}{(\zeta - P)^{k+1}} d\zeta \right) (z - P)^k$$

Moreover, since on $D(P, r_2) \setminus \bar{D}(P, r_1)$, and each $r > 0$, such that $r_1 < r < r_2$, both $|\zeta - P| = s_1$ and $|\zeta - P| = s_2$ can be continuously deformed to $|\zeta - P| = r$, so we have that

$$\sum_{k=0}^{+\infty} \left( \int_{|\zeta - P| = s_2} \frac{f(\zeta)}{(\zeta - P)^{k+1}} d\zeta \right) (z - P)^k = \sum_{k=0}^{+\infty} \left( \int_{|\zeta - P| = r} \frac{f(\zeta)}{(\zeta - P)^{k+1}} d\zeta \right) (z - P)^k$$

$$\sum_{k=-\infty}^{k=-1} \left( \int_{|\zeta - P| = s_1} \frac{f(\zeta)}{(\zeta - P)^{k+1}} d\zeta \right) (z - P)^k = \sum_{k=-\infty}^{k=-1} \left( \int_{|\zeta - P| = r} \frac{f(\zeta)}{(\zeta - P)^{k+1}} d\zeta \right) (z - P)^k$$

Hence, we conclude that

$$f(z) = \sum_{-\infty}^{+\infty} \frac{1}{2\pi i} \int_{\partial D(P, r)} \frac{f(\zeta)}{(\zeta - P)^{k+1}} d\zeta (z - P)^k$$
and the theorem is proved.

③. (Schwartz-Pick Lemma)

Let \( f : D(0, 1) \to D(0, 1) \) be holomorphic, and \( f(a) = b \) for some \( a \in D(0, 1) \), then we have the following estimation

\[
|f'(a)| \leq \frac{1 - |b|^2}{1 - |a|^2}
\]

Moreover, if \( f(a_1) = b_1 \) and \( f(a_2) = b_2 \), then

\[
\left| \frac{b_2 - b_1}{1 - b_1 b_2} \right| \leq \left| \frac{a_2 - a_1}{1 - \bar{a}_1 a_2} \right|
\]

Proof.

First, we consider the Möbius transform for any \( a \in D(0, 1) \)

\[
\phi_a(z) = \frac{z - a}{1 - \bar{a} z}
\]

Then, it is easily proved that \( \phi_a \) is biholomorphic from \( D(0, 1) \) to itself, and \( \phi_a(a) = 0 \).

Next, given that \( f(a) = b \), notice that \( \phi_{-a} \) is also holomorphic, we shall set

\[
g(z) = \phi_b \circ f \circ \phi_{-a}(z)
\]

then \( g \) is holomorphic and \( g(0) = 0 \), \( g(z) \leq 1 \) for all \( z \in D(0, 1) \). By Schwartz Lemma, we have that

\[
|g'(0)| \leq 1
\]

By the chain rule, we actually have that

\[
g'(0) = \phi'_b(b) \cdot f'(a) \cdot \phi'_{-a}(0)
\]

Also, \( \phi'_{-a}(0) = 1 - |a|^2 \), \( \phi'_b(b) = \frac{1}{1 - |b|^2} \), so we have that

\[
|f'(a)| \leq \frac{1 - |b|^2}{1 - |a|^2}
\]

Next, assume \( f(a_1) = b_1 \) and \( f(a_2) = b_2 \), then we take

\[
h(z) = \phi_{b_1} \circ f \circ \phi_{-a_1}(z)
\]

So \( h \) is again a holomorphic function with \( h(0) = 0 \) and \( h(z) \leq 1 \) for all \( z \in D(0, 1) \), and using the Schwartz Lemma gives us

\[
|h(z)| = |\phi_{b_1} \circ f \circ \phi_{-a_1}(z)| \leq |z|
\]
Put $z = \phi_{a_1}(w)$, we then have

$$|\phi_{b_1} \circ f(w)| \leq |\phi_{a_1}(w)|$$

Put $w = a_2$, and we obtain

$$|\phi_{b_1}(b_2)| \leq |\phi_{a_1}(a_2)|$$

This implies that, by the definition of $\phi_{a_1}$ and $\phi_{b_1}$,

$$\left| \frac{b_2 - b_1}{1 - b_1 b_2} \right| \leq \frac{|a_2 - a_1|}{1 - \bar{a}_1 a_2}$$

And this completes the proof of the Schwartz-Pick Lemma.

Next, we shall prove that $Aut(D(0, 1)) = \{ e^{i\theta} \frac{z - a}{1 - \bar{a}z} | \theta \in [0, 2\pi], a \in D(0, 1) \}$. To see this, first we suppose $f$ is biholomorphic from $D(0, 1)$ to $D(0, 1)$, and $f(0) = a$. Then, consider the following Möbius transform,

$$\phi_a(z) = \frac{z - a}{1 - \bar{a}z}$$

So $\phi_a \circ f(0) = 0$, and $g = \phi_a \circ f$ is biholomorphic from $D(0, 1)$ to $D(0, 1)$. Thus, the Schwartz lemma gives us

$$|g(z)| \leq |z|, \quad \text{for all } z \in D(0, 1)$$

On the other hand, since $g^{-1}$ is also biholomorphic from $D(0, 1)$ to $D(0, 1)$, with $g^{-1}(0) = 0$, so the Schwartz lemma gives us

$$|g^{-1}(w)| \leq |w|, \quad \text{for all } w \in D(0, 1)$$

Put $w = g(z)$ in the above inequality, we obtain that

$$|z| \leq |g(z)|, \quad \text{for all } z \in D(0, 1)$$

Hence, we have that $|g(z)| = |z|$ for all $z \in D(0, 1)$, and by the Schwartz lemma, we know that $g$ is just a rotation, i.e. there is a $\theta \in [0, 2\pi]$, such that

$$g(z) = \phi_a \circ f(z) = e^{i\theta} z$$

Since $\phi^{-1}_{a_1}(z) = (z + a)/(1 + \bar{a}z)$, and if we write $ae^{i\theta} = -b$, the above equality implies that our $f$ can be written as

$$f(z) = e^{i\theta} \frac{z - b}{1 - bz}$$

This proves our assumption, as $f$ is arbitrarily chosen.
(4). (Rouché’s Theorem)

Suppose \( f \) and \( g \) are both holomorphic on an open set \( U \subseteq \mathbb{C} \), and \( D(P, r) \subseteq U \). Moreover, \( f \) and \( g \) have no zeros on \( \partial D(P, r) \). If on \( \partial D(P, r) \), we have that
\[
|f(z)| < |g(z)|
\]
then \( f \pm g \) and \( g \) have the same number of zeros in \( D(P, r) \).

Next, we shall use Rouché’s Theorem to prove the following open mapping theorem.

(Open mapping Theorem)

Suppose \( f \) is holomorphic and nonconstant on a connected open set \( U \subseteq \mathbb{C} \), then \( f(U) \) is also open.

Proof.

First, for each \( z_0 \in U \), we consider \( f(z_0) \in f(U) \), and will show that there is a \( D(f(z_0), r) \) such that \( D(f(z_0), r) \subseteq f(U) \). If this is true, then \( f(U) \) is open by definition.

By the isolatedness of zeros of a nonconstant holomorphic function, we choose a suitable \( r \) such that \( f(z) - f(z_0) \) is nonzero on \( D(z_0, r) \setminus \{z_0\} \). Also, we can make that \( \delta = \min_{z \in \partial D(z_0, r)} |f(z) - f(z_0)| > 0 \).

However, for any \( z_1 \in D(z_0, r) \setminus \{z_0\} \) such that \( 0 < |f(z_1) - f(z_0)| < \delta \) (this can be done by the continuity of \( f \)), we have that
\[
|(f(z) - f(z_1)) - (f(z) - f(z_0))| = |f(z_1) - f(z_0)| < \delta \leq |f(z) - f(z_0)|,
\]
on \( \partial D(z_0, r) \)

Thus, we know that by Rouché’s theorem, \( f(z) - f(z_1) \) and \( f(z) - f(z_0) \) have the same number of zeros in \( D(z_0, r) \), hence \( f(z) - f(z_1) \) has one zero (since \( f(z) - f(z_0) = 0 \) for \( z = z_0 \)) in \( D(z_0, r) \setminus \{z_0\} \).

This means for each \( z_0 \in U \), \( \{z : 0 \leq |f(z) - f(z_0)| < \delta\} \subseteq f(U) \), which implies that \( f(U) \) is open and this completes our proof.

\( \Box \)

(5). (Liouville’s Theorem)

Suppose \( f \) is entire and \( |f(z)| \leq M \), on \( \mathbb{C} \), for some \( M > 0 \), then \( f \) is constant function.

Proof.
For each $R > 0$, we consider $h(z) = f(Rz)/M$, which is holomorphic on $D(0, 1)$ and satisfies $|h(z)| \leq 1$. So by the Schwartz-Pick Lemma, we have that for any $|z| < 1$,

$$\frac{R}{M} \cdot |f'(Rz)| = |h'(z)| \leq \frac{1 - |h(z)|^2}{1 - |z|^2} \leq \frac{1}{1 - |z|^2}$$

Next, for each $w \in \mathbb{C}$, we take enough large $R$ such that $|w|/R < 1$, then, applying above inequality with $z = (w/R)$,

$$|f'(w)| \leq \frac{M}{R} \cdot \frac{1}{1 - |w|^2/R^2}$$

Letting $R \to +\infty$, the right hand side of the above inequality goes to 0, thus $|f'(w)| = 0$ for each $w \in \mathbb{C}$, which indicates that $f$ is constant on $\mathbb{C}$.

\[\square\]

**Part B:**

1. Let

$$f(z) = \frac{z^2 - 2z + 5}{(z - 2)(z^2 + 1)}$$

Find the Laurent expansion of $f$ in

(a) the neighborhood of 2  (b) the annulus $\{ z : 1 < |z| < 2 \}$.

*Proof.*

(a). On $\{ z : 0 < |z - 2| < \sqrt{5} \}$, we can expand $f(z)$ as:

$$f(z) = \frac{z^2 - 2z + 5}{(z - 2)(z^2 + 1)}$$

$$= \frac{(z - 2)^2 + 4(z - 2) + 5 - 2(z - 2)}{(z - 2)((z - 2)^2 + 4(z - 2) + 5)}$$

$$= \frac{1}{z - 2} - \frac{2}{(z - 2)^2 + 4(z - 2) + 5}$$

$$= \frac{1}{z - 2} + i \cdot \left( \frac{1}{(z - 2) - (2 + i)} - \frac{1}{(z - 2) - (2 - i)} \right)$$

$$= \frac{1}{z - 2} + i \cdot \left( \frac{1}{2 - i} \cdot \frac{1}{1 - \frac{(z - 2)}{2 - i}} - \frac{1}{2 + i} \cdot \frac{1}{1 - \frac{(z - 2)}{2 + i}} \right)$$

$$= \frac{1}{z - 2} + \frac{i}{2 - i} \cdot \sum_{k=0}^{\infty} \left( \frac{z - 2}{2 - i} \right)^k - \frac{i}{2 + i} \cdot \sum_{k=0}^{\infty} \left( \frac{z - 2}{2 + i} \right)^k$$


(b). However, on \( \{ z : 1 < |z| < 2 \} \), we can expand \( f(z) \) as:

\[
\begin{align*}
f(z) &= \frac{z^2 - 2z + 5}{(z - 2)(z^2 + 1)} \\
&= \frac{1}{z - 2} - \frac{2}{z^2 + 1} \\
&= -\frac{1}{2} \sum_{k=0}^{\infty} \left( \frac{z}{2} \right)^k - 2 \sum_{k=0}^{\infty} (-1)^k \left( \frac{1}{z^2} \right)^{k+1}
\end{align*}
\]

\( \square \)

2. The values of the line integral

\[
\oint_{\gamma} \frac{1}{z^2(z^2 + 1)} \, dz
\]

depends on \( \gamma \)-the integration path. What are the possible values of this integral as \( \gamma \) varies over all simple closed curves?

**Proof.**

Obviously, \( f(z) = 1/z^2(z^2 + 1) \) has three poles: 0, \( i \), \( -i \). So if the interior of the closed piecewise \( C^1 \) curve \( \gamma \) does not contain any of the three poles, then \( \oint_{\gamma} \frac{1}{z^2(z^2 + 1)} \, dz = 0 \).

If the interior of \( \gamma \) contains just 0, then by generalized Cauchy’s integral formula:

\[
\begin{align*}
\oint_{\gamma} \frac{1}{z^2(z^2 + 1)} \, dz &= \oint_{\gamma} \frac{1/(z^2 + 1)}{z^2} \, dz \\
&= 2\pi i \left( \frac{1}{z^2 + 1} \right)' \big|_{z=0} \\
&= 2\pi i \frac{-2z}{(z^2 + 1)^2} \big|_{z=0} \\
&= 0
\end{align*}
\]

If the interior of \( \gamma \) contains just \( i \), then by generalized Cauchy’s integral formula:

\[
\begin{align*}
\oint_{\gamma} \frac{1}{z^2(z^2 + 1)} \, dz &= \oint_{\gamma} \frac{1/[z^2(z + i)]}{z - i} \, dz \\
&= 2\pi i \frac{1}{z^2(z + i)} \big|_{z=i} \\
&= -\pi
\end{align*}
\]
If the interior of $\gamma$ contains just $-i$, then by generalized Cauchy’s integral formula:

$$\oint_{\gamma} \frac{1}{z^2(z^2 + 1)} \, dz = \oint_{\gamma} \frac{1}{z [z^2(z - i)]} \, dz$$

$$= 2\pi i \frac{1}{z^2(z - i)} \bigg|_{z=-i}$$

$$= \pi$$

Next, if the interior of $\gamma$ contains just 0 and $i$, then by generalized Cauchy’s integral formula:

$$\oint_{\gamma} \frac{1}{z^2(z^2 + 1)} \, dz = \oint_{\gamma} \left( \frac{1}{z^2} - \frac{1}{z^2 + 1} \right) \, dz$$

$$= 0 - 2\pi i \frac{1}{z + i} \bigg|_{z=i}$$

$$= -\pi$$

Similarly, if the interior of $\gamma$ contains just 0 and $-i$, then by generalized Cauchy’s integral formula:

$$\oint_{\gamma} \frac{1}{z^2(z^2 + 1)} \, dz = \pi$$

Lastly, if the interior of $\gamma$ contains 0, $i$ and $-i$, then by generalized Cauchy’s integral formula:

$$\oint_{\gamma} \frac{1}{z^2(z^2 + 1)} \, dz = \oint_{\gamma} \left( \frac{1}{z^2} + \frac{1}{2i z + i} - \frac{1}{2i z - i} \right) \, dz$$

$$= 0 + 2\pi i \frac{1}{2i} - 2\pi i \frac{1}{2i}$$

$$= 0$$

In sum, the possible values of integral over all the simple curves are $\pi$, $-\pi$ and 0.

3. Evaluate the integral

$$\oint_{|z|=1} z^{407} \cos(1/z) \, dz$$

where the integration curve is the unit circle with its usual counter-clockwise orientation.

Proof.
On \( \{ z : 0 < |z| < \infty \} \), the function \( \cos(1/z) \) has the following Laurent expansion:

\[
\cos(1/z) = \sum_{k=0}^{\infty} \frac{(-1)^k (1/z)^{2k}}{(2k)!}
\]

This means

\[
z^{407} \cos(1/z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{407-2k}}{(2k)!}
\]

So the residue of \( f(z) = z^{407} \cos(1/z) \) at the essential singularity \( z = 0 \) is easily seen as

\[\text{Res}_f(0) = \frac{1}{408!}\]

Hence, by the residue formula, since the interior of \( \{ z : |z| = 1 \} \) contains only the only singularity of \( f, z = 0 \), and the winding number of the curve about \( z = 0 \) is 1, we have

\[
\oint_{|z|=1} z^{407} \cos(1/z) \, dz = 2\pi i \text{Res}_f(0) = \frac{2\pi i}{408!}
\]

4. Use the residue theorem to prove that

\[
\int_0^{\infty} \frac{x^2}{1 + x^4} \, dx = \frac{\pi \sqrt{2}}{4}
\]

**Proof.**

We consider the following auxiliary holomorphic function and the contour \( \gamma_R = \gamma_R^1 + \gamma_R^2 \):

\[
f(z) = \frac{z^2}{1 + z^4}
\]
Next, we parameterize $\gamma_R^1$, and $\gamma_R^2$ as:

$$\gamma_R^1(t) := \text{Re}^{2\pi it}, \quad t \in [0, \frac{1}{2}]$$

$$\gamma_R^2(t) := t, \quad t \in [-R, R]$$

On the other hand, since for each $R$, the interior $\gamma_R$ contains two poles of $f(z)$, $z_1 = -\sqrt{2}/2 + i\sqrt{2}/2$, $z_2 = \sqrt{2}/2 + i\sqrt{2}/2$, both of order 1; also, the winding number about each pole is one, so by the residue formula, we have

$$\oint_{\gamma_R} f(z) \, dz = 2\pi i \left( \text{Res}_{z_1} f(z) + \text{Res}_{z_2} f(z) \right)$$

Calculating the residues gives us

$$\text{Res}_{z_1} f(z) = \lim_{z \to z_1} \frac{z^2(z - z_1)}{1 + z^4} = -\frac{1}{4} \left( \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right)$$

$$\text{Res}_{z_2} f(z) = \lim_{z \to z_2} \frac{z^2(z - z_2)}{1 + z^4} = -\frac{1}{4} \left( -\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right)$$

So, $\oint_{\gamma_R} f(z) \, dz = \frac{\sqrt{2}\pi}{2}$ (*).

For the integral on the left hand side, we the following evaluation:

$$\oint_{\gamma_R} f(z) \, dz = \oint_{\gamma_R^1} f(z) \, dz + \oint_{\gamma_R^2} f(z) \, dz$$
And
\[
\left| \oint_{\gamma_1} f(z)dz \right| = \left| \oint_{\gamma_1} \frac{z^2}{1+z^4}dz \right|
\]
\[
\leq \oint_{\gamma_1} \frac{|z^2|}{1+z^4}dz
\]
\[
\leq \pi R \frac{R^2}{R^4 - 1}
\]
\[
\to 0, \quad \text{when } R \to +\infty
\]

\[
\oint_{\gamma_2} f(z)dz = \oint_{\gamma_R} \frac{z^2}{1+z^4}dz
\]
\[
= \int_{-R}^{+R} \frac{t^2}{t^4 + 1} dt
\]
\[
\to \int_{-\infty}^{+\infty} \frac{t^2}{t^4 + 1} dt, \quad \text{when } R \to +\infty
\]
\[
= 2 \int_{0}^{+\infty} \frac{t^2}{t^4 + 1} dt
\]

Hence, letting \( R \to +\infty \) in the left hand side of equation (*), we have
\[
2 \int_{0}^{+\infty} \frac{t^2}{t^4 + 1} dt = \frac{\sqrt{2\pi}}{2}
\]
i.e. \( \int_{0}^{\infty} \frac{x^2}{1+x^4}dx = \frac{\sqrt{2\pi}}{4} \)

5. Let \( f \) be entire and \( 0 < r < R < \infty \) be two fixed numbers. Consider the family \( \mathcal{F} \) of functions \( f_n(z) = f(nz) \) for \( z \in \{ z | r < |z| < R \} \). Assume the family \( \mathcal{F} \) is normal (in the extended sense). What can you conclude for \( f \)?

\textbf{Proof.}

We claim that \( f \) is a polynomial or constant, and the proof is as follows:

First, if there is a subsequence of \( \mathcal{F} \), say \( \{f_{n_k}\} \), which diverges uniformly on compact subsets of \( \{ z | r < |z| < R \} \), then, choose two numbers \( r_1, r_2 \) such that \( r < r_1 < r_2 < R \), and consider the compact set \( \{ z | r_1 \leq |z| \leq r_2 \} \subseteq \{ z | r < |z| < R \} \), on which \( \{f_{n_k}\} \) diverges uniformly to \( \infty \). So given any \( M > 0 \), we can find a \( n_{k_0} \), such that for all \( n_k \geq n_{k_0} \),
\[
|f_{n_k}| \geq M
\]
This implies that for all $n_k \geq n_{k_0}$, on sets of $E_{n_k} = \{ z | n_k r_1 \leq |z| \leq n_k r_2 \}$, we have

$$|f(z)| \geq M$$

Take $U = \bigcup_{k=k_0}^{\infty} E_{n_k}$, then $|f(z)| \geq M$ on $U$.

Now, if there exist $k_1 \geq k_0$ such that $\{ z | n_{k_1} r_2 < |z| < n_{k_1+1} r_1 \} \not\subseteq U$, then, by the maximal modulus theorem (we need to apply the maximal modulus theorem to $1/f$), as $|f(z)| \geq M$ on the boundaries $|z| = n_k r_2$ and $|z| = n_{k+1} r_1$, we should have $|f(z)| \geq M$ on $\{ z | n_k r_2 < |z| < n_{k+1} r_1 \}$.

Thus, we have showed that on the infinite annulus $\{ z | |z| \geq n_{k_0} r_1 \}$, $|f(z)| \geq M$.

So, actually we have

$$\lim_{z \to \infty} |f(z)| = +\infty$$

This implies that $\infty$ is a pole for $f(z)$, so the power series expansion for $f(z)$ can only have finitely many terms, i.e. $f(z)$ is polynomial.

Second, if there is a subsequence of $F$, say $\{ f_{n_k} \}$, which converges uniformly to $\tilde{f}(z)$ on compact subsets of $\{ z | r < |z| < R \}$, then again take the compact set $\{ z | r_1 \leq |z| \leq r_2 \} \subseteq \{ z | r < |z| < R \}$, and $\{ f_{n_k} \}$ converges uniformly to $\tilde{f}(z)$ on it.

Thus, since $|\tilde{f}(z)| \leq M$ on $\{ z | r_1 \leq |z| \leq r_2 \}$, we know that for large enough $n_k$, $|f_{n_k}(z)| \leq 2M$ on $\{ z | r_1 \leq |z| \leq r_2 \}$. Next, by the similar argument as above, as well as the maximal modulus theorem, we have that $|f(z)| \leq 2M$ on $\{ z | |z| \geq n_{k_0} r_1 \}$, which means that $f$ is bounded on $\mathbb{C}$, so we conclude that $f$ is constant.

In sum, $f$ can be either polynomial or constant, which depends on uniform divergence or uniform convergence.