Riemann Surfaces

Well, a Riemann surface is a certain kind of Hausdorf space. You know what a Hausdorff space is, don't you? Its also compact, ok. I guess it is also a manifold. Surely you know what a manifold is. Now let me tell you one nontrivial theorem, the Riemann-Roch Theorem

—Gian-Carlo Rota's recollection of Lefschetz lecturing in the 1940's, quoted in A Beautiful Mind by Sylvia Nasar.

Most of the theory about complex analysis we covered last semester are local theory, i.e. it works on an oepn set $U \subset \mathbf{C}$. In this seemster, we want to extend the theory to a more general object: i.e. a Riemann surface which locally just looks like an open set in $\mathbf{C}$ (but globally we can introduce the concept of holomorphic (or meromorphic) functions on it). The main topics is thus to study the relations on holomorphic or meromorphic functions. Note that, on a compact RS, holomorphic functions are constants, so for compact Riemann surfaces, it is more interesting to study meromorphic functions, or more general meromorphic differential forms. Even the mere existence of nonconstant meromorphic functions is an important and nontrivial result. Our goal is to introduce and prove the Riemann-Roch Theorem for compact Riemann surfaces (about the existence of meromorphic functions, or more general meromorphic differential forms, or more general holomorphic sections for line bundles). We introduce divisors on compact Riemann surfaces as a device for describing the zeros and poles of meromorphic functions and differentials on M. Associated to each divisor are vector spaces of meromorphic functions and differentials. The Riemann-Roch theorem is a relation between the dimensions of these spaces.

1 Results from complex analysis (previous semester)

- The Riemann sphere $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$: The following objects are (simplest) Riemann surfaces: The complex plane $\mathbf{C}$ and the Riemann sphere $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. The most important property of a Riemann surface is that one can deal with holomorphic functions on it. In a neighborhood of each point one can choose a *local coordinate* which is a one-to one correspondence between the neighborhood and an open subset of $\mathbf{C}$.

  For instance, the point $\infty \in \hat{\mathbf{C}}$ has a local coordinate $w = 1/z$. A function on a RS is holomorphic if it is presented locally ((that is, in each local coordinate) by an analytic function. For instance, a function $f : \hat{\mathbf{C}} \to \mathbf{C}$ is holomorphic if its restriction to $\mathbf{C}$ is holomorphic, as well as its restriction to $\hat{\mathbf{C}} - \{0\}$ written in the coordinate $w = 1/z$. Note: the notion is obviously good, but gives nothing for $\hat{\mathbf{C}}$: any holomorphic function on the Riemann sphere is constant.

Recall:

Theorem. (Liouville). *A bounded holomorphic function on* $\mathbf{C}$ *is constant.*

This implies non-existence of nonconstant holomorphic functions on $\hat{\mathbf{C}}$: any such function would be bounded as a function on $\mathbf{C}$, and, therefore, constant. This is a very general phenomenon: there are no nonconstant holomorphic functions on compact Riemann surfaces (can be proved by the maximum principle).

- **Meromorphic functions**. To have something interesting to study, we have to work with more general functions. These are, for instance, meromorphic functions. Recall that a meromorphic function on X is just a holomorphic function on a complement $X - S$ of X to a discrete subset S having at most poles at the points of S.

  Recall that any isolated singular point x of a function f is of one of the following three types: a removable singularity (Laurent series has no negative terms). In this case f extends to a function holomorphically at x; a pole (the Laurent series at x has finite number of negative terms); an essential singularity.

  A holomorphic function on X can be viewed as a (holomorphic) map $f : X \to \mathbf{C}$. Furthermore, a meromorphic function on X can be viewed as a holomorphic map $f : X \to \hat{\mathbf{C}}$. This differs a meromorphic function from a function having an essential singularity: the latter can not be extended to the singular point in no sense.

  Very often the amount of meromorphic functions on a Riemann surface is not very big and not very small — that is is worth studying.

- **Meromorphic functions on $\hat{\mathbf{C}}$**. The result below is well-known from the Complex Variable course.

  Lemma. Meromorphic functions on $\hat{\mathbf{C}}$ are just the rational functions in z.

  Proof. The rational functions are certainly meromorphic. Let f be a meromorphic function on $\hat{\mathbf{C}}$. The set of poles of a meromorphic function is discrete, therefore, finite since $\hat{\mathbf{C}}$ is compact. If $z_1, \dots, z_n \in \mathbf{C}$ are the poles of f of degrees $d_1, \dots, d_n$ then $p := f \prod_{i=1}^{n} (z - z_i)^{d_i}$ has no poles on $\mathbf{C}$ and has at most a pole at ∞. This means that p is a polynomial, so that f is a rational function.

- One of important general questions we will solve in the course: how does this result generalizes to general Riemann surfaces?

  Even the mere existence of nonconstant meromorphic functions is an important and nontrivial result. In this course we will prove that any Riemann surface X admits a nonconstant meromorphic function.

- Analytic continuation. Historically, the first Riemann surfaces have been defined as Riemann surfaces of a germ of analytic function. Recall that any power series $f = \sum_{n=0}^{\infty} a_i (z - z_0)^n$ having a nonzero radius of convergence $r > 0$ defines an analytic function on the disc $\{z \mid |z - z_0| < r\}$. For any point z_1 in this disc one can rewrite this function as a power series of $z - z_1$ with the radius of convergence at least $r - |z - z_1|$ — but may be greater than that. This allows one to extend an analytic function defined initially by a power series to a domain greater then the original disc.

  One proceeds as follows. We call a function element a pair (D, f) where D is a disc and f a power series around the center of D whose radius of convergence is (at least) the radius of D. A function element (D_1, f_1) is called an (immediate) analytic continuation of (D, f) if the center of D_1 belongs to D and the analytic functions f_1 and f coincide in $D_1 \cap D$. Let $\gamma : [0, 1] \to \mathbf{C}$ be a continuous path in $\mathbf{C}$ with $a = \gamma(0)$ and $b = \gamma(1)$. Analytic continuation of a function element (D, f) along γ is a collection $t \mapsto (D_t, f_t)$ of analytic elements such that D_t is a disc with center at $\gamma(t)$ and for any $s < t$ such that $\gamma([s, t]) \subset D_s$ the pair (D_t, f_t) is an immediate analytic continuation of (D_s, f_s). One can easily prove (using compactness of a segment and continuily of the radius of convergence of f_t) that any analytic continuation along a path can be equally acomplished by a finite sequence of immediate analytic continuations.

  Let now (D, f) be an analytic element and let $\{(D_\alpha, f_\alpha)\}$ be the collection of all analytic elements which can be obtained from (D, f) by analytic continuation. We can assign to (D, f) the following space: it is the quotion of the disjoint union of D_α by the equivalence relation generated by pairs of immediate analytic continuations (or in terms of the language of "germs" as follows: Let $G = \cup D_\alpha \subset \mathbf{C}$ and $a \in G$. We say that $(U, f) \sim (V, g)$ if there is an open neighborhood $W \subset U \cap V$ of a such that $f|_W = g|W$. The class of equivalence of (U, f) is denoted by f_a (is called the germ of holomorphic function f at a). The Riemann surface assign to (D, f) is the collection of all germs).

  The resulting space is locally isomorphic to $\mathbf{C}$; this is what is called the Riemann surface of (D, f).

  In this way one constructs Riemann surfaces of some standard multivalued functions. For example, consider $w = \sqrt{z}$. If $z = re^{i\phi}$, then $w = \pm\sqrt{r}e^{i\phi/2}$. We start from the function element (D, f) where $D = D(1, 1/2)$, $f(re^{i\phi}) = r^{1/2}e^{i\phi/2}$. The Riemann surface can be visualized as follows: In real analysis we were able to choose a positive value of the root; no this becomes impossible as there is no notion of "positivity" for complex numbers. However, one choose a branch of the function on any simply connected domain U of $\mathbf{C}$ not containing the origin. In particular, one can choose $U = \mathbf{C} - \mathbf{R}_{\leq 0}$. and agree for $z = re^{i\phi}$ with $\phi \in (-\pi, \pi)$ that $w = \sqrt{r}e^{i\phi/2}$. One can go further and choose another copy $U_{-} = \mathbf{C} - \mathbf{R}_{\leq 0}$ and agree to define $w = -\sqrt{r}e^{i\phi/2}$. Note that the values of w at U are not compatible when we approach to positive real semi-line from above or from below. The same is true for values of w at U_{-}. However, the values of w at U above the semi-line is compatible to the values of w at U_{-} below the semi-line. Thus, if we glue U with U_{-} so that the upper part of U is glued to the lower part of U_{-} and vice versa, we will get a univalent function w on the union of U with U_{-} (and with the real positive semi-line $\mathbf{R}_{>0}$). This is an example of what one can get gluing together series expansions at different discs.

2 Riemann Surfaces

- ## Definition

An n-dimensional complex manifold M is a Hausdorff paracompact topological space with local coordinate covering $\{U_i, \Phi_i\}$ such such

(1) Each U_i is an open subset of M and $\cup U_i = M$,

(2) $\Phi_i : U_i \to U_i^0$ is a homeomorphism from U_i onto an open subset $U_i^0 \subset \mathbf{C}^m$,

(3) If $U_i \cap U_j \neq \emptyset$, then $\Phi_i \circ \Phi_j^{-1} : \Phi_j(U_i \cap U_j) \to \Phi_i(U_i \cap U_j)$ are holomorphic.

$\phi : U \to \mathbf{C}$ is called a (coordinate) chart.

A Riemann surface is a (connected) complex manifold of dimension one.

Obvious examples: The complex plane $\mathbf{C}$ is the first example of a RS. Its only chart is $U = \mathbf{C}$ with the identity map to $\mathbf{C}$. The Riemann sphere $\hat{\mathbf{C}}$ is the first example of a compact RS. Its atlas can be built from two charts: $U_0 = \hat{\mathbf{C}} - \infty = \mathbf{C}$ and Φ_0 is the id, $U_1 = \hat{\mathbf{C}} - \{0\}$ and $\Phi_1(z) = 1/z$ if $z \neq \infty$ and $\Phi_1(\infty) = 0$. Then $\Phi_0 \circ \Phi_1^{-1} : \mathbf{C}^* \to \mathbf{C}^* : \Phi_0 \circ \Phi_1^{-1}(z) = 1/z$. Teh sphere $\Sigma = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ is also a compact RS where $U_0 = \Sigma - \{$north pole$\}$, $U_1 = \Sigma - \{$south pole$\}$, $\Phi_1(p_1, p_2, p_3) = \frac{p_1 + ip_2}{1 - p_3}, \Phi_0(p_1, p_2, p_3) = \frac{p_1 - ip_2}{1 - p_3}$, $\Phi_0 \circ \Phi_1^{-1} : \mathbf{C}^* \to \mathbf{C}^* : \Phi_0 \circ \Phi_1^{-1}(z) = 1/z$.

More examples: (1) Complex projective space: $\mathbf{P}^1(\mathbf{C}) := \mathbf{C}^2 - \{0\}/\sim$ where (z_1, z_2) is equivalent to (w_1, w_2) if and only if $(w_1, w_2) = \lambda(z_1, z_2)$. Let $U_1 = [1, z_2]$, $\phi_1 : U_1 \to \mathbf{C}$ by $[1, z_2] \mapsto z_2$ and $U_2 = [z_1, 1]$, $\phi_2 : U_2 \to \mathbf{C}$ by $[z_1, 1] \mapsto z_1$.

(2) Complex Torus: $X = \mathbf{C}/\Lambda$. Let $\omega_1, \omega_2 \in \mathbf{C}$ be $\mathbf{R}$-linear independent. Consider the lattice $\Lambda := \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$. The set Λ is a discrete subgroup of the additive group $\mathbf{C}$, so the quotient space $X := \mathbf{C}/\Lambda$ is well-defined. We define $U \subset X$ to be open if $\pi^{-1}(X) \subset \mathbf{C}$ is open, where $\pi : \mathbf{C} \to X$ is the natural projection. To find the chart, we notice that L is discrete, so there is $\epsilon > 0$ such that $\Lambda \cap \{z \in \mathbf{C} \mid |z| < \epsilon\} = \{0\}$. We define for every $z \in \mathbf{C}$ the map $\pi_z : D(z, \epsilon) \to \pi(D(z, \epsilon))$, and we leave it as an exercise to show that the maps π_z are homeomorphisms and $\mathcal{A} := \{\phi_z := \pi_z^{-1} \mid z \in \mathbf{C}\}$ is a coordinate covering. Another (equivalent) way to do is: Let $P \in \mathbf{C}/\Lambda$, the neighborhood U_P of P is the set $U_p \subset \mathbf{C}$ such that every two distinct points which are different modulo Λ. Let $\Phi_P : U_P \to \mathbf{C}$ be the identity map. Then $\{(U_P, \Phi_P) : P \in \mathbf{C}/\Lambda\}$ is a coordinate covering for X.

Example (Riemann Surface of germs of Holomorphic functions)(see the discussion above): The first Riemann surfaces have been defined as Riemann surfaces of a germ of analytic function. Let $a \in \mathbf{C}$ and consider the family of all pairs (U, f) with $a \in U$ and f being holomorphic on U. We say that $(U, f) \sim (V, g)$ if there is an open neighborhood $W \subset U \cap V$ of a such that $f|_W = g|W$. The class of equivalence of (U, f) is denoted by f_a (is called the germ of holomorphic function f at a). Let $\mathcal{O}_a$ be the set of all germs of holomorphic functions at a. The set $\mathcal{O}_a$ has a natrual structure of principal local ring where the unique maximal ideal is $M_a = \{f_a \in \mathcal{O}_a \mid f_a(a) = 0\}$. The residue field $\mathcal{O}_a/M_a$ is isomorphic to $\mathbf{C}$.

Consider now $\mathcal{O} = \cup_{a \in \mathbf{C}} \mathcal{O}_a$ and we introduce in it a topology as follows: Let (U, f) be a representative of $f_a \in \mathcal{O}_a$. Denote by $N(U, f) = \{f_z \in \mathcal{O}_z, z \in U\}$ and let a set $\Omega \in \mathcal{O}$ be open iff it contains a subset of the form $N(U, f)$. The topological space $\mathcal{O}$ becoems a Hausdorff space. In $\mathcal{O}$ there is a natrual continuous map $p : \mathcal{O} \to \mathbf{C}, p(f_a) = a$. If $N(U, f)$ is a neightborhood of f_a then $p : N(U, f) \to U$ is a homeomorphism.

Now let a fixed $f_a \in \mathcal{O}_a$, consider the collection of all analytic elements which can be obtained from f_a. Then $\mathcal{O}$ forms a Riemann surface, which is called the Riemann surface of the germ f_a. We have discussed such surafce associated to the function (germ) $\sqrt{z}$ on $D(1, 1/2)$.

3 Functions on Riemann Surfaces

- Holomorphic functions on RS. Note that the notions of harmonic and sub-harmonic functions can also be extended to the RS. Let X any Y be two

RS. A continuous map $f : X \to Y$ is called *a holomorphic map* (and we usually will not consider other maps between RS) if for each pair of charts $\phi : U \to \mathbf{C}, \psi : U \to \mathbf{C}$, the composition $\psi \circ f \circ \phi^{-1}$ is holomorphic. A holomorphic map $f : M \to \mathbf{C}$ is called *a holomorphic function*. Note that the notions of harmonic and sub-harmonic functions can also be extended to the RS.

(The sheaf of germs of Holomorphic functions): Let M be an Riemann surface. Let $a \in M$. We say that $(U, f) \sim (V, g)$ if there is an open neighborhood $W \subset U \cap V$ of a such that $f|_W = g|W$. The class of equivalence of (U, f) is denoted by f_a (is called the germ of holomorphic function f at a). Let $\mathcal{O}_a$ be the set of all germs of holomorphic functions at a. Consider now $\mathcal{O}_M = \cup_{a \in M} \mathcal{O}_a$ and we introduce in it a topology as follows: Let (U, f) eb a representative of $f_a \in \mathcal{O}_a$. Denote by $N(U, f) = \{f_z \in \mathcal{O}_z, z \in U\}$ and let a set $\Omega \in \mathcal{O}$ be open iff it contains a subset of the form $N(U, f)$. The topological space $\mathcal{O}_X$ becoems a Hausdorff space. In $\mathcal{O}_M$ there is a natrual continuous map $p : \mathcal{O}_X \to X, p(f_a) = a$. If $N(U, f)$ is a neightborhood of f_a then $p : N(U, f) \to U$ is a homeomorphism. Then $\mathcal{O}_M$ is a Riemann surface and $p : \mathcal{O}_M \to M$ is a holomorphic map.

Properties of holomorphic functions extend to manifolds:

(1) If M and N are Riemann surfaces (or complex manifolds) with M connected and $f, g : M \to N$ are holomorphic and coincide on a set with a limit point, then $f = g$ on M. Consider the set of points in which f, g coincide in a neighbourhood. It is open (automatic). It is closed (given a sequence $\{z_k\}$ its tail lies in one chart). It is not empty, for it contains the limit point; so f, g must coincide everywhere on M.

(2) Suppose M is connected and f is holomorphic on M if $|f|$ has a relative maximum, it is constant. If $|f|$ has a relative maximum, in a neighbourhood, it coincides with the constant function, use part (1).

From the maximal principle, every holomorphic map on a compact RS must be constant. As a result, meromorphic functions on a compact RS is more interesting.

- **Meromorphic functions and meromorphic differential**. In last semester, we defined the notion of *pole* and *essential singularity* for a holomorphic function on a punctured neighborhood. These notions are invariant under local biholomorphic functions and thus extends to Riemann surfaces: let f be holomorphic in a punctured neighborhood of a point $p \in M$. We say that f has a removable singularity (resp. a pole of order n) if there exists a complex chart $\phi : U \to U^0$ such that $f \circ \phi^{-1}$ has a removable singularity (resp. a pole of order n) at $\phi(p)$.

Theorem. *Let f be holomorphic in a punctured neighborhood of a point $p \in M$. (a) If f is bounded, then f has a removable singularity. (b) If $\lim_{z\to p} |f(z)| = +\infty$, then f has a pole.*

Definiton. *Let $W \subset M$ be an open subset. We say a function f on W is meromorphic at $p \in W$ if f is is holomorphic on a punctured neighborhood of a point p and has either a pole or a removable singularity at p. The function $f : M \to \mathbf{C}$ is said to be a meromorphic function if there exists a discrete set $\{p_i\} \subset M$ such that $f : M\backslash\{p_j\}_{j=1}^{\infty} \to \mathbf{C}$ is holomorphic and f is meromorphic at each pj.*

We now have a well-defined notion of a meromorphic function on a RS and also a well-defined notion of order, which is denoted by $ord_p(f)$ (note: $ord_p(f) = k$ if p is a zero of f order k, and $ord_p(f) = -k$ if p is a pole of f order k).

Example: Consider the torus $X = \mathbf{C}/\Lambda$. We define a meromorphic function $\mathcal{P} : \mathbf{C} \to \mathbf{C}$ as follows:

$$\mathcal{P}(z) := \frac{1}{z^2} + \sum_{0\neq\omega\in L} \left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right).$$

Ignoring issues of convergence, observe that $\mathcal{P}(z+\omega) = \mathbf{P}(z)$ for all $\omega \in L$, thus $\mathcal{P}$ determined a unique meromorphic function on X, which (both) is called the Weierstrass $\mathcal{P}$-function.

- **The Argument Principle**.

Complex Differential Forms and Meromorphic 1-forms: Recall that in the theory of complex analysis (last semester) when we study the the Argument Principle (or the residue theorem), we used f'/f. However, for a meromorphic function f on a RS M, the notion of f' is no longer making sense (since, for a chart (U_α, ϕ_α), f'_α (where $f_\alpha = f \circ \phi_\alpha^{-1}$, is only a local object). However, if we use, instead the notion: $f'_\alpha dz_\alpha$ where $z_\alpha = \phi_\alpha$, then, by the chain rule: $f'_\alpha dz_\alpha = f'_\beta dz_\beta$, where $z_\beta = \phi_\beta = (\phi_\beta \circ \phi_\alpha^{-1}) \circ \phi_\alpha$. We call df a differential one-form on M (which is the exterior derivative of f). The one-form is usally used to be integrated over paths. For example, we define $\int_\gamma \frac{df}{f}$ (as we extend the argument principle to RS) as follows : Let γ be a smooth curve in M, let $\phi_j : U_j \to V_j$, $j = 1, \ldots, N$ be N coordinate charts such that $\gamma \subset \cup_{j=1}^N U_j$. For each $W_j \subset\subset U_j$ such that $\gamma \subset \cup_{j=1}^N W_j$, and let ψ_j be a smooth function with $\psi_j \equiv 1$ on W_j and whose support is in U_j. Put $\chi_j := \left(\sum_{k=1}^N \psi_k\right)^{-1} \psi_j$. We define

$$\int_\gamma \frac{df}{f} := \sum_{j=1}^N \int_{\phi_j(\Gamma \cap U_j)} \chi_j(\phi_j^{-1}(z)) \frac{(f \circ \phi_j^{-1})'(z)}{f \circ \phi_j^{-1}(z)} dz.$$

It is a standard exercise in advanced calculus to show that the definition is independent of the choice of the function χ_j. Moreover, if we choose different coordinate charts, say $\tilde{\phi}_j : \tilde{U} \to \tilde{V}$, and let $\tilde{z}$ be the corresponding variable in $\tilde{V}$ then on $V \cap \tilde{V}$,

$$\begin{aligned}
\frac{(f \circ \phi_j^{-1})'(z)}{f \circ \phi_j^{-1}(z)} dz &= \frac{(f \circ \phi_j^{-1} \circ (\phi \circ \tilde{\phi}^{-1})'(\tilde{\phi} \circ \phi^{-1}(z))}{f \circ \tilde{\phi}^{-1}(\tilde{\phi} \circ \phi_j^{-1}(z))} d(\tilde{\phi} \circ \phi^{-1}(z)) \\
&= \frac{(f \circ \phi_j^{-1})'((\phi \circ \tilde{\phi}^{-1} \circ \tilde{\phi} \circ \phi^{-1}(z))}{f \circ \tilde{\phi}^{-1}(z)} d(\phi \circ \tilde{\phi}^{-1}(\tilde{\phi} \circ \phi^{-1}(z)) \\
&= \frac{(f \circ \phi_j^{-1})'(z)}{f \circ \phi_j^{-1}(z)} dz,
\end{aligned}$$

where the last two equalities follow from the chain rule. Thus the definition is well-posed.

In general, a meromorphic 1-form ω on a RS M is an assignment, for every local coordinate (U, z_U), $\omega = f_U dz_U$, where f_U is meromorphic, and for every (U, z_U) and (W, z_W), on $U \cap W$, we have

$$f_U \frac{dz_U}{dz_W} = f_W.$$

Integration of $\int_\gamma \omega$ is defined in a similar manner as above.

In general

- A 0-form on M is a function on M.

- A 1-form ω is an (ordered) assignment, for every local coordinate (U, z_U), $\omega = f_U dz_U + g_U d\bar{z}_U$, where f_U and g_U are two (local) functions, and is invariant under coordinate change, i.e. and for every (U, z_U) and (W, z_W), on $U \cap W$, we have $\omega = f_U dz_U + g_U d\bar{z}_U = f_W dz_W + g_W d\bar{z}_W$.
- A 2-form Ω is an assignment, for every local coordinate (U, z_U), $\Omega = f_U dz_U \wedge d\bar{z}_U$, where f_U is a (local) function, and is invariant under coordinate change. Here we used the "exterior" multiplication of forms. This multiplication satisfies the following: $dz \wedge dz = 0, dz \wedge d\bar{z} = -d\bar{z} \wedge dz, d\bar{z} \wedge d\bar{z} = 0$.
- If f is a C^1 function on M, then $df := \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}$ is a 1-form. d is called the *exterior* operator. The $d\omega$ for any 1-form ω is defined in a similar manner.
- Write $\partial f := \frac{\partial f}{\partial z} dz$. Then $d = \partial + \bar{\partial}$. Let $\triangle = 2i\partial\bar{\partial}$, which is called the Laplace operator.

Integration of Forms:

- **Integration of 1-form**. Let γ be piecewise smooth curve in M, and ω be a smooth 1-form on M. Let $\{(U_\alpha, \phi_\alpha\}_{\alpha \in A}$ be a collection of local coordinates, and divide γ into

$$\gamma = \gamma_1 \cup \cdots \cup \gamma_m,$$

where $\gamma_j : [t_{j-1}, t_j] \to$ with $0 = t_0 < t_1 < \cdots < t_m = 1$ such that every image of γ_j is contained one of the given coordinate neighborhood. Assume that γ_j is contained in U_α and let $\omega = u_\alpha dz_\alpha + v_\alpha d\bar{z}_\alpha$. Define

$$\int_{\gamma_j} \omega := \int_{t_{j-1}}^{t_j} \left(u_\alpha(\phi_\alpha \circ \gamma) \frac{d\phi_\alpha \circ \gamma}{dt} + v_\alpha(\phi_\alpha \circ \gamma) \frac{d\overline{\phi_\alpha \circ \gamma}}{dt} \right) dt.$$

This is well-defined, which is independent of the choice of the coordinate neighborhood, since if γ_j is contained in U_β, then $u_\alpha dz_\alpha + v_\alpha d\bar{z}_\alpha = u_\beta dz_\beta + v_\beta d\bar{z}_\beta$. We define

$$\int_\gamma \omega := \sum_{j=1}^m \int_{\gamma_j} \omega.$$

Note that it is independent of the division of γ into small arcs, since if

$$\gamma = \gamma_1' \cup \cdots \cup \gamma_l',$$

Then we can put the division points together to get a new division of γ, and it is easy to see both integral values of the two old sum is equal to the new one.

– **Integration of 2-form**. Let Ω is a smooth two form and U_α be a coordinate neighborhood. Write $\Omega = G_\alpha dz_\alpha \wedge d\bar{z}_\alpha$. Assume D is a domain in M which is covered by U_α, then we define

$$\int_D \Omega := \int_{\phi_\alpha(D)} G_\alpha(z_\alpha) dz_\alpha \wedge d\bar{z}_\alpha.$$

his is well-defined, which is independent of the choice of the coordinate neighborhood, since if D is contained in U_β, then $G_\beta(z_\beta) = G_\alpha(z_\alpha)|f'_{\alpha\beta}|^2$ where $f_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$ is the transition function. Thus

$$\int_{\phi_\alpha(D)} G_\alpha(z_\alpha) dz_\alpha \wedge d\bar{z}_\alpha = \int_{\phi_\beta(D)} G_\alpha(f_{\alpha\beta})|f'_{\alpha\beta}|^2 dz_\beta \wedge d\bar{z}_\beta = \int_{\phi_\beta(D)} G_\beta(z_\beta) dz_\beta \wedge d\bar{z}_\beta.$$

We can extend the above definition to the domain which is covered by finitely many coordinate neighborhoods.

– **Stokes formula**: Let ω be a 1-form, D is a closed domain with smooth boundary, then

$$\int_{\partial D} \omega = \int_D d\omega.$$

Residues: Let $\omega = f dz$ be a meromorphic 1-form, and $p \in M$ be a pole of ω. Define $\text{res}_p\omega := res_p(f)$, it is easy to check that the definition is independent of the choice of the coordinate. Alternatively, for a small disc D centered at p,

$$res_p(\omega) = \frac{1}{2\pi i}\int_{\partial D} \omega.$$

Theorem (residue theorem). *If M is compact, then for any meromorphic 1-form ω,*

$$\sum_{p\in M} res_p\omega = 0.$$

Proof. Note that since M is compact, the above sum is only a finite sum.

Assume $p_1, \ldots, p_k$ are poles of ω. Let B_j be the small discs containing p_j only and mutually disjoint. Let $M' = M - \cup_{j=1}^k B_j$, then ω is holomorphic on M', so $d\omega = 0$ on M'. From the Stoke's theorem,

$$0 = \int_{M'} d\omega = \int_{\partial M'} \omega = -\sum_{j=1}^k \int_{\partial B_j} \omega.$$

Hence

$$\sum_{p \in M} res_p \omega = 0.$$

Theorem. *Let M be RS and $D \subset M$ be an open subset whose closure is compact and whose boundary is piecewise smooth. If f is meromorphic on M with no zeros or poles on ∂D, then*

$$\frac{1}{2\pi} \int_{\partial D} \frac{df}{f} = \sum_{x \in D} ord_x(f).$$

Proof. The proof is similar to above by using $\omega = df/f$ which is a meromorphic 1-form. The set

$$A := \{x \in D \mid ord_x(f) \neq 0\}$$

is locally finite and thus finite. Let $p_1, \ldots, p_k$ ne the points of A and $B_j, 1 \leq j \leq k$ the small discs containing p_j only and mutually disjoint in D and let $E = D - \cup_{j=1}^{k} B_j$, then ω is holomorphic on M', so $d\omega = 0$ on E'. From the Stoke's theorem,

$$0 = \int_E d\omega = \int_{\partial E} \frac{df}{f} = \int_{\partial D} \frac{df}{f} - \sum_{j=1}^{k} \int_{\partial D_j} \frac{df}{f}.$$

On the other hand, by the argument principle (from last semester),

$$\frac{1}{2\pi i} \int_{\partial D_j} \frac{df}{f} = ord_{p_j}(f).$$

This completes the proof.

Corollary. *Let M be a compact RS and f be meromorphic on M, then*

$$\sum_{x \in M} ord_x(f) = 0.$$

- ## Holomorphic mappings between Riemann Surfaces:

A meromorphic function f on M can be viewed as a holomorphic mapping $f : M \to \mathbf{P}^1$. Thus, it is important to study the properties for general holomorphic mappings between RS.

Let X any Y be two RS. A continuous map $f : X \to Y$ is called *a holomorphic map* (and we usually will not consider other maps between RS) if for each pair of charts $\phi : U \to \mathbf{C}, \psi : U \to \mathbf{C}$, the composition $\psi \circ f \circ \phi^{-1}$ is holomorphic.

Theorem (Normal Form Theorem). *Let $F : X \to Y$ be a holomorphic map between two RSs, and $x \in X$. Then there exist two coordinate charts $\phi_1 : U_1 \to V_1, \phi_2 : U_2 \to V_2$ at x and $F(x)$ respectively and a unique integer $m = m_x$ (which is called the multiplicity) such that $\phi_1(x) = \phi_2(F(x)) = 0$ and*

$$\phi_2 \circ F \circ \phi_1^{-1}(z) = z^m.$$

Proof. Choose any pair of coordinate charts. After translation, we assume that $\tilde{\phi}_1(x) = \phi_2(F(x)) = 0$. Then $\phi_2 \circ F \circ \tilde{\phi}_1^{-1}(\zeta) = \zeta^m e^{h(\zeta)}$. Let $\psi(\zeta) := \zeta e^{\frac{1}{m}h(\zeta)}$ which is locally 1-1. Let $\phi_1 := \psi \circ \tilde{\phi}_1$. This will serve our purpose.

Definition. *(1) We call $m := Mult_x(F)$ the multiplicity of F at $x \in X$.*

(2) If $Mult_x(F) \geq 2$, we say that F is ramified at x and that x is a ramification point for F.

(3) If $p \in X$ is a ramification point for F, we call $F(p)$ a branch poitn of F.

- **Degree of a holomorphic map**.

  Theorem. *Let $F : X \to Y$ be a holomorphic map between two connected compact RSs. Then*
  $$\deg(F) := \sum_{x \in F^{-1}(y)} Mul_x(F)$$
  is independent of y.

- Riemann-Hurwitz Formula:

  Definition. *Let M be a compact RS (regarded as a manifold of real-dimension 2) with smooth boundary (possibly empty),*

  (1) A 0-simplex, or vertex, is a point. A 1-simplex, or edge, is a set homeomorphic to a closed interval. A 2-simplex, or face, is a set homeomorphic to the triangle $\{(x, y) \in [0, 1] \times [0, 1]; x + y \leq 1\}$.

  (2) A triangulation of M is a decomposition of M into faces, edges and vertices, such that the intersection of any two faces is a union of edges and the intersection of any two edges is a union of vertices.

(3) Ket M have a triangulation with total number of faces equal to F, total number of edges equal to E, and total number of vertices equal to V. The number $\chi(M) := F - E + V$ is independent of the choices of the triangulation, which is called the Euler characteristic of M. $\chi(M) := 2 - 2g$ where g is called the genus of M.

Theorem (Riemann-Hurwitz formula). *$F : X \to Y$ be a holomorphic map between two connected compact RSs. Then*

$$2g(X) - 2 = \deg(F)(2g(Y) - 2) + \sum_{x \in X} (Mult_x(f) - 1).$$

Proof. Let $d = \deg(f)$. Take a triangulation of Y such that every branch point is a vertex. (There may, of course, be other vertices). Suppose this triangulation has F faces, E edges, V_u unbranched verticies, and V_b branched vertices.

Since the preimage of every unbranched point has d points, we obtain a traingulation of X with dF faces, dE edges and W vertices. To express W in terms of V and f, we observe that if $x \in X$ is a ramification point for f, then $Mult_x(f)$-many points are collapsed into one point, so that we have

$$W = dV - \sum_{y \in V_b} \sum_{x \in f^{-1}(y)} Mult_x(f) - 1 = dV - \sum_{x \in X} (Mult_x(f) - 1).$$

The last equality follows because $Mult_x(f) = 1$ for all unramified points x. This proves the theorem.

- Automorphism groups of Complex Tori:

Let $M = \mathbf{C}/\Lambda$, where $\Lambda := \mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$, and $\omega_1, \omega_2 \in \mathbf{C}$ are $\mathbf{R}$-linear independent.

Theorem. *$f : \mathbf{C}/\Lambda_1 \to \mathbf{C}/\Lambda_2$ is a biholomorphic map if and only if there exists $F(z) = az + b$ with $a \neq 0$ such that F maps the equivalent classes w.r.t Λ_1 to equivlanent classes w.r.t. Λ_2.*

The proof uses the lifting property (for universal coverings) from $f : \mathbf{C}/\Lambda_1 \to \mathbf{C}/\Lambda_2$ to get $F : \mathbf{C} \to \mathbf{C}$ and use the following result proved in last semester: If $F \in Aut(\mathbf{C})$ then $F = az + b$.

Corollary. *$\mathbf{C}/\Lambda_1$ is biholomorphic to $\mathbf{C}/\Lambda_2$ iff there exists $a \neq 0$ such that $F(z) = az$ sends an equivalent class with respect to Λ_1 to the equivalent class with respect to Λ_2.*

Hence,

$$a\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = F\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}\omega_1'\\ \omega_2'\end{pmatrix},$$

and

$$F^{-1}\begin{pmatrix}\omega_1'\\ \omega_2'\end{pmatrix} = B\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}.$$

Thus

$$\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = F^{-1}\circ F\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = F^{-1}(A\begin{pmatrix}\omega_1'\\ \omega_2'\end{pmatrix} = AF^{-1}\begin{pmatrix}\omega_1'\\ \omega_2'\end{pmatrix} = AB\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}.$$

Since ω_1 and ω_2 are real-linealy independent, $AB = I$. Hence $det(A)\det(B) = 1$. Since entries of A and B are integers, $det(A) = \pm 1$. Let $\tau = \omega_1/\omega_2$, $\tau' = \omega_1'/\omega_2'$. Then we have

Theorem. *Let* $\Lambda = Span_{\mathbf{Z}}\{1,\tau\}, \Lambda' = Span_{\mathbf{Z}}\{1,\tau'\}$, *with* $Im\tau, Im\tau' > 0$. *Then* $\mathbf{C}/\Lambda$ *is biholomorphic to* $\mathbf{C}/\Lambda'$ *if and only if*

$$\tau' = \frac{a_{11}\tau + a_{12}}{a_{21}\tau + a_{22}}, \qquad (*)$$

where $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbf{Z}$ *and* $a_{11}a_{22} - a_{12}a_{21} = 1$.

We now introduce an equivalent relation as follows: $\mathbf{C}/\Lambda_1 \sim \mathbf{C}/\Lambda_2$ iff $\mathbf{C}/\Lambda_1$ is biholomorphic to $\mathbf{C}/\Lambda_2$, and denote by $\mathcal{A}_1$ the set of equivalent classes. So, from the theorem, $\Lambda_1 = \{1,\tau\}, \Lambda_2 = \{1,\tau'\}$, then they below to the same equivalent class if and only if (*) is satisfied. To desciribe clearly about $\mathcal{A}_1$. W econisder $H = \{\tau \in \mathbf{C} \mid Im(z) > 0\}$ the upper-half plane on $\mathbf{C}$. Then (*) defines a map

$$\tau \mapsto \tau' = \frac{a_{11}\tau + a_{12}}{a_{21}\tau + a_{22}},\ a_{11}a_{22} - a_{12}a_{21} = 1.$$

The set of such transformation becomes a group, and is denoted by $SL(2,\mathbf{Z})$ (called the modular group). We now define the fundamental domain $D \subset H$ of the modular group as the subset such that (i) every $\tau \in H$ is congruent to $\tau' \in D$ mod $SL(2,\mathbf{Z})$, (ii) Any two distinct points in D are not congruent mod $SL(2,\mathbf{Z})$.

A modular function is a holomorphic function or a meromorphic function defined on H which is invariant udner the action of the group $SL(2,\mathbf{Z})$.

4 Harmonic Forms

Harmonic Forms and Holomorphic Forms:

- $f \in C^2$ is called a *harmonic function* if $\partial\bar{\partial} f = 0$.
- A 1-from $\omega \in C^1$ is called a *harmonic form* if locally we can write $\omega = df$ where f is harmonic.
- We introduce the star-operator: for ant 1-form $\omega = fdz + gd\bar{z}$, $\quad \star\omega := -ifdz + igd\bar{z}$.
- **Theorem**: *A 1-form ω is harmonic if and only if ω is closed and is co-closed, i.e. $d\omega = 0$ and $d(\star\omega) = 0$.*
- A 1-form ω is called a *holomorphic form* (resp. meromorphic) if locally $\omega = fdz$ where f is holomorphic (resp. meromorphic). A meromorphic 1-form is also called a *abel form*.
- **Theorem**: *If f is harmonic, then $\omega := \partial f$ is holomorpihc.*
- **Theorem**: *A 1-form ω is holomorphic if and only if $\omega = h + i \star h$ where h is a harmonic 1-form.*

Weyl Lemma:

- **Weyl Lemma**. *Let $D(0, R) = \{z \in \mathbf{C} \mid |z| < R\}$. Then $\phi \in L^2(D)$ is a harmonic function if and only if*

$$\int_D \phi \triangle \eta = 0, \quad \forall \eta \in C_0^\infty(D).$$

Remarks (1) This is indeed the homework problem #64 on Page 240 of the Krantz's book.

(2) The part "$\Longrightarrow$" part is easy to prove. The t "$\Longleftarrow$" part is also easy to prove if we assume that, in addition, u is C^2. Indeed, from Stoke's theorem,

$$\int_D \phi \triangle \eta - \int_D \eta \triangle \phi = 0.$$

Hence

$$\int_D \eta \triangle \phi = 0$$

for all $\eta \in\in C_0^\infty(D)$. This implies that $\triangle\phi = 0$.

Proof of the Weyl Lemma. For any given $\epsilon > 0$, choose a real-valued C^∞ function $\rho(r), r \in [0, +\infty)$ such that $\rho_\epsilon(r) \equiv 1$ for $r \in [0, \epsilon/2)$, $\rho_\epsilon(r) \equiv 0$ for $r \in (\epsilon, \infty)$, and $0 \leq \rho_\epsilon(r) \leq 1$ on $[\epsilon/2, \epsilon]$. Let

$$\Omega_\epsilon(r) = \frac{1}{\pi i}\rho_\epsilon(r)\log r.$$

For any function $\mu \in C_0^\infty(D)$, consider the function

$$\eta_\epsilon(\xi) = \int_{\mathbf{C}} \Omega_\epsilon(|z - \zeta|)\mu(z) dz \wedge d\bar{z}.$$

When ϵ is small enough, η_ϵ has compact support. On the other hand, we can write it as

$$\eta_\epsilon(\xi) = \int_{\mathbf{C}} \Omega_\epsilon(|z|)\mu(z + \xi) dz \wedge d\bar{z}.$$

Hence η_ϵ is smooth, and

$$\frac{\partial^2}{\partial\bar{\xi}}\eta_\epsilon(\xi) = \int_{\mathbf{C}} \Omega_\epsilon(|z - \xi|)\frac{\partial}{\partial\bar{z}}\mu(z) dz \wedge d\bar{z}$$

$$\frac{\partial}{\partial\xi}\eta_\epsilon(\xi) = \int_{\mathbf{C}} \Omega_\epsilon(|z - \xi|)\frac{\partial}{\partial z}\mu(z) dz \wedge d\bar{z}.$$

We claim that

$$\frac{\partial^2}{\partial\xi\partial\bar{\xi}}\eta_\epsilon(\xi) = -\mu(\xi) + \int_{\mathbf{C}} \frac{\partial^2}{\partial\xi\partial\bar{\xi}}\Omega_\epsilon(|z - \xi|)\mu(z) dz \wedge d\bar{z}.$$

To prove the claim, fix $\xi_0 \in D$, and write

$$\eta_\epsilon(\xi) \equiv f(\xi) + g(\xi),$$

where ξ satsifies $|\xi - \xi_0| < \epsilon/4$ and

$$f(\xi) = \frac{1}{\pi i}\int_{|z-\xi_0|<\epsilon/4} \mu(z)\ln|z - \xi| dz \wedge d\bar{z}$$

$$g(\xi) = \frac{1}{\pi i}\int_{|z-\xi_0|>\epsilon/4} \Omega_\epsilon(|z - \zeta|)\mu(z) dz \wedge d\bar{z}.$$

It is easy to check that

$$\frac{\partial^2 f}{\partial \bar{\xi}} = -\mu(\xi).$$

When $|\xi - \xi_0| < \epsilon/4$ and $|z - \xi_0| < \epsilon/4$, $|\xi - z| < \epsilon/2$. Hence $\Omega_\epsilon(|z-\zeta|) = \ln|z-\xi|$ $(z \neq \xi)$, and is harmonic in ξ. Theorefore,

$$\begin{aligned}\frac{\partial^2 g}{\partial \bar{\xi}} &= \int_{|z-\xi_0|>\epsilon/4} \frac{\partial^2}{\partial \bar{\xi}} \Omega_\epsilon(|z-\zeta|)\mu(z) dz \wedge d\bar{z} \\ &= \int_{\mathbf{C}} \frac{\partial^2}{\partial \bar{\xi}} \Omega_\epsilon(|z-\zeta|)\mu(z) dz \wedge d\bar{z}.\end{aligned}$$

This proves the claim. Assuming the claim holds, then, using $\eta = \eta_\epsilon$ the assumption gets

$$\begin{aligned} 0 &= \frac{1}{2i}\int_D \phi \triangle \eta_\epsilon \\ &= -\int_D \mu(\xi)\phi(\xi) d\xi \wedge d\bar{\xi} + \int_D \phi(\xi) d\xi \wedge d\bar{\xi} \int_{\mathbf{C}} \frac{\partial^2 \Omega_\epsilon(|z-\xi|)}{\partial \xi \partial \bar{\xi}} \mu(z) dz \wedge d\bar{z} \\ &= -\int_{\mathbf{C}} \mu(\xi)\left[\phi(\xi) - \int_D \phi(z) \frac{\partial^2 \Omega_\epsilon(|z-\xi|)}{\partial z \partial \bar{z}} dz \wedge d\bar{z}\right] d\xi \wedge d\bar{\xi}. \end{aligned}$$

Since μ is arbitray, we get

$$\phi(\xi) = \int_D \phi(z) \frac{\partial^2 \Omega_\epsilon(|z-\xi|)}{\partial z \partial \bar{z}} dz \wedge d\bar{z}.$$

When $|\xi - z| < \epsilon/2$,

$$\frac{\partial^2 \Omega_\epsilon(|z-\xi|)}{\partial z \partial \bar{z}} = 0,$$

hence

$$\phi(\xi) = \int_{D \setminus \triangle_{\epsilon/2}} \phi(z) \frac{\partial^2 \Omega_\epsilon(|z-\xi|)}{\partial z \partial \bar{z}} dz \wedge d\bar{z}.$$

Thus $\phi(\xi)$ is smooth. We have proved, in the remark, that if *phi* is C^2, then it is harmonic. This finishes the proof.

Hilbert Space of 1-forms:

- A measurable 1-form is called square-integrable if

$$\|\omega\|^2 := \int_M \omega \wedge \star\bar{\omega} < +\infty.$$

- On $L^2(M)$, we introduce an inner product

$$(\omega_1, \omega_2) := \int_M \omega_1 \wedge \star\bar{\omega_2}.$$

  $L^2(M)$ becomes an Hilbert space under this inner product.

- Let E be the closure in $L^2(M)$ of the set $\{df \mid f \in C_0^\infty(M)\}$, and E^* be the closure in $L^2(M)$ of the set $\{\star df \mid f \in C_0^\infty(M)\}$. We have

$$L^2(M) = E \bigoplus E^\perp, \quad L^2(M) = E^* \bigoplus E^{*\perp}.$$

  It is not hard to verify that

$$E^\perp = \{\omega \in L^2(M) \mid (\omega, df) = 0, \quad f \in C_0^\infty(M)\},$$

$$E^{*\perp} = \{\omega \in L^2(M) \mid (\omega, \star df) = 0, \quad f \in C_0^\infty(M)\},$$

- **Theorem**.*Let $\omega \in L^2(M) \cap C^1(M)$. Then*

  (i) $\omega \in E^{*\perp}$ if and only if ω is closed.

  (ii) $\omega \in E^\perp$ if and only if ω is co-closed.

- If ω is smooth (i.e. C^1), then ω is harmonic if and only if $\omega \in E^\perp \cap E^{*\perp}$.

- The Weyl lemma allows to remove the condition of "smoothness" in above, i.e.

  Theorem. *$\omega \in E^\perp \cap E^{*\perp}$ is the set of ALL harmonic forms (note, he definition of harmonic form requires C^1).*

- From the definition, $E \subset E^{*\perp}$ and $E^* \subset E^\perp$, Thus elements in E and E^* are always orthogonal to each other. Thus

$$L^2(M) = E \bigoplus E^* \bigoplus (E \bigoplus E^*)^\perp = E \bigoplus E^* \bigoplus (E^\perp \bigoplus E^{*\perp}).$$

  This proves

  Theorem.

$$L^2(M) = E \bigoplus E^* \bigoplus H$$

  where H is the set of all harmonic 1-forms.

The decomposition theorem for smooth differential forms:

From above, $L^2(M) = E \bigoplus E^* \bigoplus H$, so every $\omega \in L^2(M)$ can be written as

$$\omega = \alpha + \beta + h, \quad \alpha \in E, \beta \in E^*, h \in H.$$

However, we need more information about α and β.

- **Lemma**. *If $\omega \in E \cap C^1$, then ω is exact. If $\omega \in E^* \cap C^1$, then ω is co-exact.*

  The proof involves the the construction of the one-form defined by a closed curve: Let γ be a simple closed curve in M. Choose an open neighborhood Ω of γ so that $\Omega - \gamma = \Omega^+ \cup \Omega^-$ is a disjoint union of two annuli. Choose a smaller neighborhood Ω_0 and its two parts $\Omega_0^\pm = \Omega_0 \cap \Omega^\pm$. Choose an orientation of γ so that Ω^- is to the left of γ. Choose a real-valued function f smooth on $M - \gamma$ such that $f|_{\Omega_0^-} = 1, \quad f|_{M-\Omega^-} = 0$. We define a one-form η_γ by the formula

  $$\eta_\gamma|_{\Omega-\gamma} = df, \quad \eta_\gamma|_{\Omega_0} = 0.$$

  The form η_γ is obviously closed and smooth.

- **Theorem (decomposition theorem)**. *Let $\omega \in L^2(M) \cap C^1(M)$, then there exists C^2 functions f and g such that*

  $$\omega = df + \star dg + h, \quad df \in E, \star dg \in E^*, h \in H.$$

Existence Theorems:

- **Theorem**. *Let $p \in M$, then for any positive integer n, there exists a meromorphic 1-form ω on M with p as a pole of order $n+1$.*

  Proof. Consider $p \in U_0 \subset U_1 \subset M$. Take $\rho \in C^\infty(M)$ with $\rho = 1$ on U_0 and $\rho = 0$ on $M \backslash U_1$. Let z be a local coordinate in U_1 with $z? = 0$. Let

  $$w := \left(-\frac{\rho}{nz^n}\right)$$

  and $\psi := dw$. Notice that

  $$\psi := d\left(-\frac{\rho}{nz^n}\right) = \left(-\frac{\rho_z}{nz^n} + \frac{\rho}{z^{n+1}}\right) dz - \frac{\rho_{\bar{z}}}{nz^n} d\bar{z}.$$

  The $(0,1)-$part of ψ is smooth on M (so $\psi - i \star \psi$ is smooth on M), thus $\psi - i \star \psi = df + \star dg + h$ with h harmonic. Consider $\alpha := \psi - df = dw - df = \star dg + i \star dw + h$. This means tha it is closed and c-closed on $M \backslash \{p\}$. Hence it is harmonic on $M \backslash \{p\}$.

5 Duality and the bilinear relations

- Recall from the Stokes theorem, if ω is closed and γ_0, γ_1 are homotopic paths, then
$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega.$$

- Homology: It follows from above stokes' theorem that a closed 1-form ω defines a map
$$[\gamma] \in \pi_1(X) \mapsto \int_\gamma \omega \in \mathbf{C}.$$
It is clear from the properties of integrals that this map is a homomorphism, and thus, since $\mathbf{C}$ is an abelian group, the kernel of this map must contain the commutation subgroup of $\pi_1(X)$. Define the quotient group
$$H_1(M, \mathbf{Z}) := \pi_1(X)/[\pi_1(X), \pi_1(X)].$$
It is called the first homology group of the surface (it is a free-abelian group). This the map extends to
$$[\gamma] \in H_1(X) \mapsto \int_\gamma \omega \in \mathbf{C}.$$
Note that this group can also be realized as a quotient of the Abelian group of closed 1-chains modulo boundary of 2-chains, i.e.

Alternative definition of $H^1(M, \mathbf{Z})$: Fix a triangulation on M. Its vertices are called the 0-simplices, its edges are called 1-simplices, every small triangles (faces) are called 2-simplices. WLOG, we assume that all sides of the small triangles are piecewise smooth. The orientation of the surface induces the orientation on the triangles which in turn can be used to orient the edges bounding the triangles. An edge receives opposite orientations from the two triangles for which it is a common edge. Furthermore, the vertices $\{p_1, \ldots, \}$ can be used to label wedges and triangles. Thus $< p_1, p_2 >$ is the oriented edge from p_1 to p_2, ... An n-chain is a finite combination of n-simplices with integer coefficients. We define an operator ∂ from n-chains to $(n-1)-$chains as follows: For $n = 1$, $\delta < p_1, p_2 >= p_2 - p_1$, $n = 2$, $\partial < p_1, p_2, p_3 >=< p_2, p_3 > - < p_1, p_3 > + < p_1, p_2 >$. The preceding defines ∂ on n-simplex and extends to n-chains. Define $H_n(M, \mathbf{Z}) = Z_n/B_n$, where $Z_n = Ker\partial$, and $B_n = Im\partial$. Any closed oriented continuous curve γ can be deformed homotopically into a 1-cycle in the triangulation (see [Springer] Springer, G., Introduction to Riemann Surfaces, Chelsea Publishing Co., New York (1981) for detail).

From Stokes theorem, let D be a 2−chain with piecewise smooth boundary, then

$$\int_D d\omega = \int_{\partial D} \omega.$$

Hence, for a closed form ω, the map

$$[\gamma] \in H_1(X) \mapsto \int_\gamma \omega \in \mathbf{C}$$

is again well-defined.

- **De-Rham Cohomology**. Let $\Lambda_1(M)$ be the set of smooth closed 1-form on M. Two elements $\omega_1, \omega_2 \in \Lambda_1(M)$ is said to be equivalent if $\omega_1 - \omega_2$ is $d-$exact. Denote by $[\omega]$ the equivalent class of $[\omega]$. The group of the collection of all such equivalent classes is called the *de Rahm* cohomology, and is denoted by $H^1_{DR}(M)$. From the decomposition theorem, $H^1_{DR}(M) = H(M)$.

- **Pairing of H_1 and H^1_{DR}**: Let ω be a smooth form, for a piecewise differential path γ, we define

$$(\gamma, \omega) := \int_\gamma \omega.$$

Let α, β be two closed 1−chain such that $\alpha - \beta$ is exact, i.e. $\alpha - \beta = \partial D$, then from Stokes' theorem

$$(\alpha - \beta, \omega) = \int_{\partial D} \omega = \int_D d\omega,$$

so if ω is closed, then $(\alpha - \beta, \omega) = 0$. On the other hand, if $\omega_1 - \omega_2$ is exact, then $(\alpha, \omega_1 - \omega_2) = 0$ since α is closed, so

$$\int_\alpha [\omega] = \int_\alpha \omega.$$

Thus (γ, ω) gives a paring on $H_1(M, \mathbf{Z}) \times H^1_{DR}(M) \to \mathbf{C}$. We'll show that this pairing is non-degenerate, i.e. it induces an isomorphism $H_1(M, \mathbf{Z}) = (H^1_{DR}(M))^*$, or equivalently it satisfies that if $(\gamma, \omega) = 0$ for all d-closed ω, then $[\gamma] = 0$, and if $(\gamma, \omega) = 0 = 0$ for all $[\gamma] \in H^1(M, \mathbf{Z})$, then $\omega = 0$. To do so, we recall:

- **Uniformization Results**: According to the uniformization theorem, every compact orientable 2-real dimensional manifold is hemeomorphic to g-torus (g is called the genus of M) with $g \geq 0$. We wish now to use the standard presentation of a compact R.S. of genus g. For $g = 0$, it is holomeomorphic to a sphere which is simply connected. For $g > 0$, M can be obtained from a $4g$-gon by indentification of the edges defined by the word

$$x_1 y_1 x_1^{-1} y_1^{-1} \cdots x_g y_g x_g^{-1} y_g^{-1}.$$

With the common vertex of the sides as a base point, one shows that $\pi_1(M)$ is generated by the simple loops $a_1, \ldots, a_g$ and $b_1, \ldots, b_g$ corresponding to the edges x_i and y_i, subject to one relation

$$\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1.$$

Hence the holomogy group $H_1(M)$ is free abelian group on the generators $[a_j], [b_j], j = 1, \ldots, g$. In particular, $H_1(M) = \mathbf{Z}^{2g}$.

- Intersection pairing on H_1 (Intersection Theory on Compact Riemann Surfaces): Define a pairing (intersection number) on the set of cycles on M by the formula

$$a \cdot b = \int_M \eta_a \wedge \eta_b = (\eta_a, - \star \eta_b).$$

(recall that η_γ are real so we do not need complex conjugation). The exterior product of differential form is dual to the intersection number.

Proposition. *The intersection pairing satisfies the following properties.*

1. *The intersection $a \cdot b$ depends only on the homology classes of a and b.*
2. *One has $a \cdot b = -b \cdot a$.*
3. *$a \cdot b \in \mathbf{Z}$. In case the intersection points of the curves a and b are transversal, $a \cdot b$ is the (signed) number of intersection points.*

Proof. The first property has already been explained: intergrals of a closed form along homotopic paths are the same. The second property results from the anticommutativity of the multiplication of one-forms.

The third property can be checked for simple closed curves since any piecewise smooth closed curve is a finite union of simple closed curves. In this case $a \cdot b = \int_a \eta_b$ and we have to check that each intersection point of $a$with b contributes 1 or -1, depending on the orientation of the curves at the intersection point. Recall that η_b is defined as differential of a function f_b having a discontinuity along b. The function f_b is zero far away from b. Thus, the integral over a can be presented as a sum of the integrals over small segments of a_i of a containing the intersection points x_i of a with b.

The intergral $\int_{a_i} \eta_b$ has been already calculated once. The result was 1 or -1. This finishes the proof.

From above, we know that the pairing so defined counts the number of times a intersects b. This will imply, in particular, that

$$a_i \cdot a_j = b_i \cdot b_j = 0, \quad a_i \cdot b_i = 1, \quad a_i \cdot b_j = 0 \quad (i \neq j).$$

We thus deduce that the intersection matrix in the basis $(a_1, \ldots, a_{2g})$ of $H_1(M)$ looks like

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \quad (*)$$

In what follows we will work with any fixed basis of $H_1(M)$ having the same intersection matrix. We will can such basis a *canonical basis* of $H_1(M)$. We do not care whether this basis comes from a polygonal presentation of M.

- The dimension $\dim_{\mathbf{C}} H^1_{DR}(M)$.

Theorem. *Let M be a compact RS with genus g, then* $\dim_{\mathbf{C}} H^1_{DR}(M) = \dim_{\mathbf{C}} H(M) = 2g$.

Proof. Let $[\gamma_1], \ldots, [\gamma_{2g}]$ be a canonical basis of $H_1(M)$. Consider

$$\Psi : H^1(M) \to \mathbf{C}^{2g} : [\omega] \mapsto \left(\int_{\gamma_1} \omega, \cdots, \int_{\gamma_{2g}} \omega \right).$$

It is a linear transformation. If $\dim_{\mathbf{C}} H^1_{DR}(M) > 2g$, there there is a non-trivial ω (assumed to be harmonic) such that $\Psi(\omega) = \mathbf{0}$. Since, as remarked above, Any closed oriented continuous curve γ can be deformed homotopically into a 1-cycle in the triangulation, this implies that $\int_\gamma \omega = 0$ for any closed path γ. Then $\omega = df$ where f is the function defined by the formula

$$f(x) = \int_{z_0}^{z} \omega.$$

The formula defines a single-valued function. The function f is automatically harmonic since $d\star df = d\star\omega = 0$. Since M is compact, there are no nonconstant harmonic functions by the maximim principle. Thus $\omega = 0$ contradicts with our selection. Hence $\dim_{\mathbf{C}} H^1_{DR}(M) \leq 2g$.

On the other hand, let $\omega_j = \eta_{\gamma_{g+j}},\ \omega_{g+j} = -\eta_{\gamma_j}, j = 1, \ldots, g$. Then, from (*),

$$\int_{\gamma_j} \omega_k = \delta_{jk}.$$

Thus $\Psi([\omega_j]) = (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 is at the j-th place. Hence $\Psi([\omega_1]), \ldots, \Psi([\omega_{2g}])$ is a basis for $H^1_{DR}(M)$. so $\dim_{\mathbf{C}} H^1_{DR}(M) = 2g$. This proves the theorem (Note: this also shows that the pairing between $H_1(M, \mathbf{Z})$ and $H^1_{DR}(M)$ is non-degenerate).

- The numbers $\int_{a_1} \omega, \cdots, \int_{a_g} \omega$ (respectively $\int_{b_1} \omega, \cdots, \int_{b_g} \omega$) are called the a-periods (resp. b-periods) of ω. From above, if ω is harmonic and its a-periods and b-periods vanishes, then $\omega = 0$.

- **Decomposition of H**: Let H be the set of harmonic forms on M. From the previous discussion, the operator $\alpha \mapsto \frac{1}{2}(\alpha + i\star\alpha)$ transforms any harmonic form into a holomorphic form and acts identically on holomorphic forms. Its kernel consists of antiholomorphic forms since if $\alpha + i\star\alpha = 0$, one has $\bar{\alpha} - i\star\bar{\alpha} = 0$ which means that $\bar{\alpha}$ is holomorphic. We denote by $\mathcal{H}$ the space of holomorphic forms. This proves the following

Theorem. *One has a canonical decomposition*

$$H = \mathcal{H} \bigoplus \bar{\mathcal{H}}.$$

of the space of harmonic forms into the sum of holomorphic and antiholomorphic forms. In particular, $\dim_{\mathbf{C}} \mathcal{H} = g$.

Proposition (bilinear relation). *Let ω and ω' be closed one-forms on M. Then*

$$\int_M \omega \wedge \omega' = \sum_{i=1}^{g} \left(\int_{a_i} \omega \int_{a_{g+1}} \omega' - \int_{a_{g+1}} \omega \int_{a_i} \omega' \right).$$

Proof. Both expressions are bilinear ω, ω'. Both vanish of one of them is closed (Stokes). Thus, we can assume they are harmonic. it is sufficient to check the formula from $\omega = \alpha_i, \omega' = \alpha_j$. In this case the claim amount to the formula

$$(a_i, a_j) = J_{ij}.$$

Corollary. *If ω is a holomorphic 1-form, and its $a-$periods are zero, then* $\omega = 0$.

Proof. From above, by taking $\omega' = \bar{\omega}$, then

$$\|\omega\|^2 = \int_M \omega \wedge \bar{\omega} = \sum_{i=1}^{g} \left(\int_{a_i} \omega \int_{a_{g+1}} \bar{\omega} - \int_{a_{g+1}} \omega \int_{a_i} \bar{\omega} \right) = 0.$$

Hence $\omega = 0$.

- ## The bilinear relation for meromorphic differential forms

Theorem. *Let ω is a holomorphic 1-form and $\tilde{\omega}$ is a meremorphic 1-form which has only one pole at $p \in M$ with residue zero. Assume that locally*

$$\omega = (a_0 + a_1 z + \cdots)dz$$

$$\tilde{\omega} = \left(\frac{c_m}{z^m} + \cdots + \frac{c_{-2}}{z^2} + c_0 + c_1 z + \cdots \right) dz.$$

Then

$$\sum_{j=1}^{g} \left(\int_{a_j} \omega \int_{b_j} \tilde{\omega} - \int_{b_j} \omega \int_{a_j} \tilde{\omega} \right) = 2\pi i \sum_{n=2}^{m} \frac{c_{-n} a_{n-2}}{n-1}.$$

The theorem is a key to the proof of Riemann-Roch theorem.

Proof. Note that $M_0 := M \backslash \{a_1, \ldots, a_g, b_1 \ldots, b_g\}$ is simply connected, so there exissta smooth function f (defined as $f(p) = \int_{p_0}^{p} \omega$ for $p \in M_0$) such that $\omega = df$. Note that f can be extended to the bounday, but f may not have the same values on the boundary. So by Stoke's theorem, and from the Residue theorem,

$$0 = \int_M \omega \wedge \tilde{\omega} = \int_{M_0} \omega \wedge \tilde{\omega} = \int_{\partial M_0} f \tilde{\omega}.$$

On the other hand, it is easy to see that

$$\int_{a_j} f\tilde{\omega} + \int_{a_j^{-1}} f\tilde{\omega} = \int_{a_j} \left[\int_{z_0}^{z} \omega - \int_{z_0}^{z'} \omega \right] \tilde{\omega}$$

where z, z' are in a_j and a_j' and are equivalent. Notice that ω have the same values on the boundary,

$$\int_{z_0}^{z} \omega - \int_{z_0}^{z'} \omega = \int_{z'}^{z} \omega = \int_{z'}^{p_j'} \omega - \int_{b_j} \omega + \int_{p_j}^{z} \omega = -\int_{b_j} \omega,$$

where the$\int_{z'}^{p'_j} \omega$ and $\int_{p_j}^{z} \omega$ are cancelled out. Thus

$$\int_{a_j} f\tilde{\omega} + \int_{a_j^{-1}} f\tilde{\omega} = -\int_{a_j} \tilde{\omega} \int_{b_j} \omega.$$

Similarly, we can prove that

$$\int_{b_j} f\tilde{\omega} + \int_{b_j^{-1}} f\tilde{\omega} = \int_{b_j} \tilde{\omega} \int_{a_j} \omega.$$

Thus we get

$$\int_M \omega \wedge \tilde{\omega} = \sum_{j=1}^{g} \left(\int_{b_j} \tilde{\omega} \int_{a_j} \omega - \int_{a_j} \tilde{\omega} \int_{b_j} \omega \right).$$

Divisors and Riemann-Roch Theorem:

The proof of Riemann-Roch Theorem:

Proof. We first prove the Riemann inequality

$$r(D^{-1}) \geq \deg(D) - g + 1.$$

When $D = 0$, it is done. So assume that $D > 0$, i.e. $D = \sum_{j=1}^{n} \alpha_j p_j$ with $\alpha_j > 0$. Write $D_1 = \sum_{j=1}^{n} (\alpha_j + 1) p_j$ and let

$$\Omega_0(D_1^{-1}) = \{\omega \in \Omega(D_1^{-1}), res_{p_j}(\omega) = 0, \int_{a_j} \omega = 0, j = 1, \ldots, g\}.$$

The point of introducing this space that the exterior derivative d maps the space we are interested in, namely $L(D^{-1})$ into $\Omega_0(D_1^{-1})$. This follows by considering the action of d locally near the poles. If $f \in L(D^{-1})$, then the poles of *df* will have one greater than the poles of f. Moreover, all the residues of *df* will be zero as will be the integrals of *df* over any closed cycle, in particular the a_j cycles. Thus

$$d : L(D^{-1}) \to \Omega_0(D_1^{-1}).$$

and we may compute the dimension of $L(D^{-1})$ using

$$\dim L(D^{-1}) = \dim(Ker(d)) + \dim(Im(d)).$$

The dimension of $Ker(d) = 1$ since any function f with $df = 0$ is constant, and constants are contained $L(D^{-1})$ since we are assuming that D is integral. Thus

$$\dim L(D^{-1}) = 1 + \dim(Im(d)).$$

We now derive the lower bound of $\dim(Im(d))$. Let $\tau_k^{(j)}$ be a meromorphic 1-form with zero residues and zero $a-$period, holomorphic on $M\backslash\{p_j\}$ and with, locally in a local coordinate system vanishing at p_j, the the principal part in the Laurent expansion as dz/z^k, $k \geq 2$. Thus, $\tau_2^{(j)}, \ldots, \tau_{\alpha_j}^{(j)}, j = 1, \ldots, n$ are in $\Omega_0(D_1^{-1})$ and they are clearly linear independent. Thus $\dim \Omega_0(D_1^{-1}) \geq \deg(D)$.

Let $f \in L(D^{-1})$, and assume that the principal part in the Laurent expansion at p_j of df is

$$\left(\frac{c_2^{(j)}}{z^2} + \cdots + \frac{c^{(j)}}{z^{m_j}}\right) dz, \quad m_j = \alpha_j + 1.$$

Hence

$$df - \sum_{j=1}^{n} \sum_{k=2}^{m_j} c_k^{(j)} \tau_k^{(j)}$$

is a holomorphic 1-form whose $a-$periods are zero. Hence it is identically zero on M, i.e.

$$df = \sum_{j=1}^{n} \sum_{k=2}^{m_j} c_k^{(j)} \tau_k^{(j)}.$$

Because the $b-$ periods of df is also zero, the b-periods of the right sides are all zero.

Conversely, if $c_k^{(j)}$ are any given numbers, then the 1-form

$$\phi := \sum_{j=1}^{n} \sum_{k=2}^{m_j} c_k^{(j)} \tau_k^{(j)} \in \Omega_0(D_1^{-1}).$$

But ϕ may not be in $Im(d)$. In order to have $\phi \in Im(d)$, we need that all b-periods of ϕ must be all zero, i.e.

$$0 = \int_{b_l} \phi = \sum_{j=1}^{n} \sum_{k=2}^{m_j} c_k^{(j)} \int_{b_l} \tau_k^{(j)}, \quad l = 1, \ldots, g. \quad (*)$$

The above system of linear equations has g equations, $\deg(D)$ unknowns $c_k^{(j)}$.

When $\deg(D) < g$, the Riemann inequality holds automatically. When $\deg(D) > g$, then the solution space of the above system of linear equations has dimension at least $\deg(D) - g$. Hence $\dim_{\mathbf{C}}(L(D^{-1})) \geq \deg(D) - g$. Hence we proved the Riemann inequality.

We now discuss the general case of D, i.e. $D = \sum_{j=1}^{k} n_k p_k$ with n_k are integers. WLOG, we assume that $n_j > 0$ for $1 \leq j \leq l$ and other s $n_j < 0$. Let $D^+ = n_1 p_1 + \cdots + n_l p_l$. Then applying the previous result,

$$r((D^+)^{-1} \geq \deg D^+ - g + 1.$$

Also from the expansion of $f \in L((D^+)^{-1})$ at p_j $(j > l)$ we know that

$$r(D^{-1}) - \sum_{j=l+1}^{k} n_j \geq r((D^+)^{-1}),$$

hence

$$r(D^{-1}) \geq \deg(D^+) + \sum_{j=l+1}^{k} n_j - g + 1 = \deg(D) - g + 1.$$

In above, we just used the simple estimate of the $\dim(Im(d))$. To obtain the more precise version, we need to have a more precise estimate which involves the rank of the coefficient matrix, where we need the bilinear relation for meromprphic forms,

Proof of RR. We first consider the case that $D > 0$, i.e. $D = \sum_{j=1}^{n} \alpha_j p_j$ with $\alpha_j > 0$. Let $\phi_1, \ldots, \phi_g$ be a canonical basis for $\mathcal{H}(M)$ and assume that locally

$$\phi_k = (a_{k0}^{(j)} + \cdots a_{k1}^{(j)} + \cdots)dz,$$

where z is local coordinate with $z(p_j) = 0$.

Let $\phi \in \Omega(D)$ and write $\phi = \lambda_1\phi_1 + \cdots + \lambda_g\phi_g$, then $\lambda_1, \ldots, \lambda_g$ must be the solution of the following equations

$$\sum_{k=1}^{g} \lambda_k a_{kl}^{(j)} = 0, \quad 0 \leq l \leq \alpha_j - 1, \ 1 \leq j \leq n \quad (**)$$

which is a system with g unknowns and $\deg(D)$ equations. Hence $\Omega(D)$ is isomorphic to the solution space of the above equations. Assume that rank of the coefficient matrix of (**) is r, then the dimension of its solution space is $g - r$, l.e $i(D) = g - r$. Assume that rank of the coefficient matrix of (*) is r^*, then the dimension of its solution space of (*) is $\deg(D) - r^*$. Hence $r(D^{-1}) = \deg(D) - r^* + 1$. Thus it remains to show that $r = r^*$. In deed, from the bilinear relation with $\omega = \phi_l, \tilde{\omega} = \tau_k^{(j)}$, we have

$$\int_{b_l} \tau_k^{(j)} = 2\pi \left(\frac{a_{l,(k-2)}^{(j)}}{k-1} \right),$$

where $2 \leq k \leq \alpha_j + 1, 1 \leq l \leq g$. This means that if we multiply each row of the coefficient matrix of (**) with a non-zero complex number, and take a transpose, then we get the coefficient matrix of (*). This proves that $r = r^*$.

We now prove the general case for D.

Applications of Riemann-Roch Theorem:

6 Differential Goemetry of Riemann Surfaces