Riemannian Geometry: Key to Homework #1

1. Compute the first and the second fundamental form of the surface $z = f(x, y)$. (note: the first fundamental form is also called the metric).

Solution: Consider the parameterization

$$\mathbf{X} = (x, y, f(x, y)).$$

The metric induced from the standard metric in $\mathbf{R}^3$ is

$$g_{11} = \mathbf{X}_1 \cdot \mathbf{X}_1 = (1 + f_x^2),\ g_{12} = g_{21} = \mathbf{X}_1 \cdot \mathbf{X}_2 = f_x f_y,\ g_{22} = \mathbf{X}_2 \cdot \mathbf{X}_2 = (1 + f_y^2).$$

Its second fundamental form is

$$h_{11} = \frac{f_{xx}}{\sqrt{f_x^2 + f_y^2 + 1}}, \quad h_{12} = h_{21} = \frac{f_{xy}}{\sqrt{f_x^2 + f_y^2 + 1}}, \quad h_{22} = \frac{f_{yy}}{\sqrt{f_x^2 + f_y^2 + 1}}.$$

2. (a) Compute the first and the second fundamental form of the torus parameterized by

$$\mathbf{X}(u^1, u^2) = ((a+r\cos u^1)\cos u^2, (a+r\cos u^1)\sin u^2, r\sin u^1), \quad 0 < u^1 < 2\pi,\ 0 < u^2 < 2\pi.$$

(b) Find the matrix of the shape operator $S_P : T_P(M) \to T_P(M)$, where $P = \mathbf{X}(u^1, u^2)$, with respect to the basis $\{\mathbf{X}_1, \mathbf{X}_2\}$.

Solution: (a)

$$g_{11} = r^2,\ g_{12} = g_{21} = 0,\ g_{22} = (a + r\cos u)^2,$$

$$h_{11} = r,\ h_{12} = 0,\ h_{22} = (a + r\cos u)\cos u.$$

$$A = (g_{ij})^{-1}(h_{ij}) = \begin{pmatrix} \frac{1}{r} & 0 \\ 0 & \frac{\cos u}{a+r\cos u} \end{pmatrix}.$$

3. Find the Gauss curvature, mean curvature, principal curvatures and the corresponding principal directions of the following surfaces

(a) $\mathbf{X}(u^1, u^2) = (a(u^1 + u^2), b(u^1 - u^2), 4u^1u^2)$ where a and b are constant.

(b) The cylinder: $\mathbf{X}(u^1, u^2) = (a\cos u^1, a\sin u^1, u^2)$.

Solution: (a) $\mathbf{X}_1 = (a, b, 4u^2), \mathbf{X}_2 = (a, -b, 4u^1), \mathbf{X}_{11} = (0,0,0), \mathbf{X}_{22} = (0,0,0), \mathbf{X}_{12} = \mathbf{X}_{21} = (0,0,4)$. The unit normal is

$$\mathbf{n} = \frac{\mathbf{X}_1 \times \mathbf{X}_2}{\|\mathbf{X}_1 \times \mathbf{X}_2\|} = \frac{(2b(u^1+u^2), 2a(u^2-u^1), -ab)}{(4b^2(u^1+u^2)^2 + 4a^2(u^1-u^2)^2 + a^2b^2)^{1/2}}.$$

The first fundamental form is

$$g_{11} = a^2 + b^2 + 16(u^2)^2, \quad g_{12} = g_{21} = a^2 - b^2 + 16u^1u^2, \quad g_{22} = a^2 + b^2 + 16(u^1)^2.$$

The second fundamental form is

$$h_{11} = 0, \quad h_{12} = h_{21} = \frac{-4ab}{(4b^2(u^1+u^2)^2 + 4a^2(u^1-u^2)^2 + a^2b^2)^{1/2}}, \quad h_{22} = 0.$$

$$det(g_{ij}) = 16a^2(u^1-u^2)^2 + 16b^2(u^1+u^2)^2 + 4a^2b^2.$$

The Gauss curvature is

$$K = \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2} = \frac{-4ab}{(4b^2(u^1+u^2)^2 + 4a^2(u^1-u^2)^2 + a^2b^2)^3}$$

the mean curvature is

$$H = \frac{h_{11}g_{22} - 2h_{12}g_{12} + h_{22}g_{11}}{2(g_{11}g_{22} - g_{12}^2)} = \frac{ab(a^2 - b^2 + 16u^1u^2)}{(4b^2(u^1+u^2)^2 + 4a^2(u^1-u^2)^2 + a^2b^2)^{3/2}}.$$

To calculate the principal curvatures, note that

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}^{-1} = \frac{1}{g_{11}g_{22} - g_{12}^2} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{pmatrix},$$

So the matrix of the shape operator S_P with respect to the basis $\{\mathbf{X}_1, \mathbf{X}_2\}$ is

$$A = \frac{1}{(g_{11}g_{22} - g_{12}^2)^{3/2}} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{pmatrix} \cdot \begin{pmatrix} 0 & -8ab \\ -8ab & 0 \end{pmatrix} = \frac{8ab}{(g_{11}g_{22} - g_{12}^2)^{3/2}} \begin{pmatrix} g_{12} & -g_{22} \\ -g_{11} & g_{12} \end{pmatrix}.$$

Write $\mu = \dfrac{8ab}{(g_{11}g_{22} - g_{12}^2)^{3/2}}$, then

$$det(A - \lambda I) = (\mu g_{12} - \lambda)^2 - \mu^2 g_{11}g_{22}.$$

Setting $det(A - \lambda I) = 0$, i.e.

$$(\mu g_{12} - \lambda)^2 - \mu^2 g_{11} g_{22} = 0,$$

we get

$$\mu g_{12} - \lambda = \pm \mu \sqrt{g_{11} g_{22}}.$$

Hence, the eigenvalues(principal curvatures) are

$$\kappa_1 = \mu(g_{12} + \sqrt{g_{11} g_{22}}), \kappa_2 = \mu(g_{12} - \sqrt{g_{11} g_{22}}).$$

To get the principal directions, for $\kappa_1 = \mu(g_{12} + \sqrt{g_{11} g_{22}})$, we solve $(A - \kappa_1 I)\mathbf{v} = \mathbf{0}$, i.e.,

$$\begin{pmatrix} -\mu\sqrt{g_{11} g_{22}} & -\mu g_{22} \\ -\mu g_{11} & -\mu\sqrt{g_{11} g_{22}} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We get (we need normalize it to get a unit-vector!!!)

$$\begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} = \frac{1}{\sqrt{1 + \frac{g_{11}}{g_{22}}}} \begin{pmatrix} 1 \\ -\sqrt{\frac{g_{11}}{g_{22}}} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{g_{22}}{g_{11}+g_{22}}} \\ -\sqrt{\frac{g_{11}}{g_{11}+g_{22}}} \end{pmatrix}.$$

Similarily,

$$\begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{g_{22}}{g_{11}+g_{22}}} \\ \sqrt{\frac{g_{11}}{g_{11}+g_{22}}} \end{pmatrix}.$$

Hence the principal directions are

$$\mathbf{e}_1 = \sqrt{\frac{g_{22}}{g_{11} + g_{22}}}\mathbf{X}_1 - \sqrt{\frac{g_{11}}{g_{11} + g_{22}}}\mathbf{X}_2, \quad \mathbf{e}_2 = \sqrt{\frac{g_{22}}{g_{11} + g_{22}}}\mathbf{X}_1 + \sqrt{\frac{g_{11}}{g_{11} + g_{22}}}\mathbf{X}_2,$$

where $\mu = \frac{8ab}{(g_{11} g_{22} - g_{12}^2)^{3/2}}$.

(ii) By direct calculation, the first fundamental form is

$$g_{11} = a^2, g_{12} = g_{21} = 0, g_{22} = 1,$$

and the second fundamental form is

$$h_{11} = -a, h_{12} = h_{21} = 0, h_{22} = 0.$$

Hence the matrix of the shape operator S_P with respect to the basis $\{\mathbf{X}_1, \mathbf{X}_2\}$ is

$$A = \begin{pmatrix} -1/a & 0 \\ 0 & 0 \end{pmatrix}.$$

Solving the equation $det(A - \lambda I) = 0$, we get the principal curvatures $\kappa_1 = 0, \kappa_2 = -1/a$. To get the principal directions, for $\kappa_1 = 0$, we solve $(A - \kappa_1 I)\mathbf{v} = \mathbf{0}$, i.e.,

$$\begin{pmatrix} -1/a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We get (we need normalize it to get a unit-vector!!!) $\xi_1 = 0, \eta_1 = 1$, so $\mathbf{e}_1 = \mathbf{X}_2$. Similarily, for $\kappa_2 = -1/a$, we get $\xi_1 = 1, \eta_1 = 0$, so $\mathbf{e}_2 = \mathbf{X}_1$.

The Gauss curvature is $K = \kappa_1 \kappa_2 = 0$ and the mean curvature is

$$H = \frac{\kappa_1 + \kappa_2}{2} = -\frac{1}{2a}.$$

4. Find the Gauss curvature and the mean curvature for the surface

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$

Solution:

$$K = \frac{4a^6 b^6}{(a^4 b^4 + 4a^4 v^2 + 4b^4 x^2)^2};$$

$$H = \frac{a^4 b^2 (b^4 + 4y^2) + a^2 b^4 (a^4 + 4x^2)}{(a^4 b^4 + 4a^4 y^2 + 4b^4 x^2)^{3/2}}.$$