Math 6397 Riemannian Geometry,Hodge Theory on Riemannian Manifolds
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1 Hodge Theory on Riemannian Manifolds

- **Global inner product for differential forms** Let (M, g) be a Riemannian manifold. In a local coordinate $(U; x^i)$, let

$$\eta = \sqrt{G} dx^1 \wedge \cdots \wedge dx^m.$$

η in fact is a global m-form, called *the volume form of* M. We first define the inner product for differential forms. Let ϕ, ψ are two r-forms. Let (U, x^i) be a local coordinate. We write

$$\phi|_U = \frac{1}{r!} \phi_{i_1 \cdots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r},$$

$$\psi|_U = \frac{1}{r!} \psi_{j_1 \cdots j_r} dx^{j_1} \wedge \cdots \wedge dx^{j_r}.$$

We define, the inner product $< \ , \ >$ of ϕ, ψ as

$$< \phi, \psi >= \frac{1}{r!} \phi^{i_1 \cdots i_r} \psi_{i_1 \cdots i_r} = \sum_{i_1 < \cdots < i_r} \phi^{i_1 \cdots i_r} \psi_{i_1 \cdots i_r},$$

where $\phi^{i_1 \cdots i_r} = g^{i_1 j_1} \cdots g^{i_r j_r} \phi_{j_1 \cdots j_r}$. It is important to note that the definition is independent of the choice of local coordinates. We also have $< \phi, \phi > \geq 0$ and $< \phi, \phi >= 0$ if and only if $\phi = 0$.

We now define the **global** inner product of ϕ, ψ as

$$(\phi, \psi) = \int_M < \phi, \psi > \eta,$$

where η is the volume form of M.

- **The exterior differential operator d and its co-operator** Denote by $\Lambda^r(M)$ the set of smooth r-forms on M. Let $(\ , \ )$ be the (global) inner product defined above. As the formal adjoint operator of the exterior differential operator d, the *codifferential operator* $\delta : \Lambda^{r+1}(M) \to \Lambda^r(M)$ is defined by, for every $\phi \in \Lambda^r(M), \psi \in \Lambda^{r+1}(M)$,

$$(d\phi, \psi) = (\phi, \delta\psi).$$

- **Hodge-star operator**. In order to find the expression of the codifferential operator δ, we introduce the Hodge-star operator $*$, which is an isomorphism $* : \Lambda^r(M) \to \Lambda^{m-r}(M)$ defined by, for every $\phi, \eta \in \Lambda^r(M)$,
$$\phi \wedge (*\psi) =< \phi, \psi > \eta.$$
Let ω be a r-form. Let (U, x^i) be a local coordinate. We write
$$\omega|_U = \frac{1}{r!} \sum_{i_1,\ldots,i_r} a_{i_1 \cdots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r}.$$
Then
$$*\omega = \frac{\sqrt{G}}{r!(m-r)!} \delta^{1\cdots m}_{i_1 \cdots i_m} a^{i_1 \cdots i_r} dx^{i_{r+1}} \wedge \cdots \wedge dx^{i_m},$$
where
$$a^{i_1 \cdots i_r} = g^{i_1 j_1} \cdots g^{i_r j_r} a_{j_1 \cdots j_r},$$
and $\delta^{1\cdots m}_{i_1 \cdots i_m}$ is the Levi-Civita permutation symbol, i.e. $\delta^{12\cdots m}_{i_1 \cdots i_m} = 1$ if $(i_1 \cdots i_m)$ is an even permutation of $(12 \ldots m)$, $\delta^{12\cdots m}_{i_1 \cdots i_m} = -1$ if $(i_1 \cdots i_m)$ is an odd permutation of $(12 \ldots m)$, $\delta^{12\cdots m}_{i_1 \cdots i_m} = 0$ otherwise. It can be shown that $*\omega$ is is independent of the choice of local coordinates. So $*\omega$ is a globally defined $(m-r)$-form (it can be regarded as an alternative definition). The operator $*$ which sends r-forms to $(m-r)$-forms.

  It has the following properties, for any r-forms ϕ and ψ:

  (1) $\phi \wedge *\psi =< \phi, \psi > \eta$,

  (2) $*\eta = 1, *1 = \eta$,

  (3) $*(*\phi) = (-1)^{r(m+1)}\phi$,

  (4) $(*\phi, *\psi) = (\phi, \psi)$.

- **Expression of the codifferential operator δ in terms of the Hodge-Star operator**. Define $\delta = (-1)^{mr+1} * \circ d \circ * : \Lambda^{r+1}(M) \to \Lambda^r(M)$, where $\Lambda^r(M)$ is the set of smooth r-forms, is called the *codifferential operator*. It is easy to verify that $\delta \circ \delta = 0$. We also have the following very important property for δ: For $\phi \in \Lambda^r(M), \psi \in \Lambda^{r+1}(M)$, we have

$$(d\phi, \psi) = (\phi, \delta\psi),$$

i.e. δ **is conjugate to** d. So $(-1)^{mr+1} * \circ d \circ *$ is the expression of the odifferential operator δ.

Proof. Note

$$\begin{aligned} d(\phi \wedge *\psi) &= d\phi \wedge *\psi + (-1)^r \phi \wedge d(*\psi) \\ &= d\phi \wedge *\psi + (-1)^r(-1)^{mr+r} \phi \wedge *(*d * \psi) \\ &= d\phi \wedge *\psi - \phi \wedge *\delta\psi. \end{aligned}$$

Then desired identity is obtained by applying the Stokes theorem.

- **Hodge-Laplace operator.** We define the Hodge-Laplace operator

$$\tilde{\triangle} = d\delta + \delta d : \Lambda^r(M) \to \Lambda^r(M).$$

For $f \in C^\infty(M)$, then $\delta(f) = 0$, so

$$\tilde{\triangle}(f) = \delta(df) = - * d * df, \quad \tilde{\triangle} f\eta = *\tilde{\triangle} f = -d * df.$$

Let (U, x^i) be a local coordinate, then

$$df|_U = \frac{\partial f}{\partial x^i} dx^i,$$

$$\begin{aligned} *df|_U &= \frac{\sqrt{G}}{(m-1)!} \delta^{1\cdots m}_{i_1\cdots i_m} g^{i_1 j} \frac{\partial f}{\partial x^j} dx^{i_2} \wedge \cdots \wedge dx^{i_m} \\ &= \sqrt{G} \sum_{i=1}^m (-1)^{i+1} g^{ij} \frac{\partial f}{\partial x^j} dx^1 \wedge \cdots \wedge d\hat{x}^i \wedge \cdots \wedge dx^m. \end{aligned}$$

Hence

$$\begin{aligned} (\tilde{\triangle} f)\eta|_U &= -d(*df)|_U = -\frac{\partial}{\partial x^i}\left(\sqrt{G} g^{ij} \frac{\partial f}{\partial x^j}\right) dx^1 \wedge \cdots \wedge dx^m \\ &= - \triangle f\eta|_U. \end{aligned}$$

This tells us

$$\tilde{\triangle} f = - \triangle f.$$

So $-\tilde{\triangle}$ when acts on $C^\infty(M)$ is the Beltrami-Laplace operator $\triangle$.

- **Hodge Theory**. In this section, we denote the Hodge-Laplace operator by $\triangle$. Let $\mathcal{H}^r(M) = ker\triangle$ and $\mathcal{H} = \bigoplus \mathcal{H}^r(M)$. Let $\bigwedge^*(M) = \bigoplus_{r=0}^{\infty} \Lambda^r(M)$.

  The Hodge theorem *Let (M, g) be an n-dimensional compact oriented Riemannian manifold without boundary. For each integer $0 \le r \le n$, $\mathcal{H}^r(M)$ is finite dimensional, and there exists a bounded linear operator $G : \bigwedge^*(M) \to \bigwedge^*(M)$ (called Green's operator) such that*

  (a) $kerG = \mathcal{H}$;

  (b) G keeps types, and commute with the operators $*, d$ and δ;

  (c) G is a compact operator, i.e. the closure of image of an arbitrary bounded subset of $\bigwedge^*(M)$ under G is compact;

  (d) $I = \mathcal{H} + \triangle \circ G$, where I is the identity operator, and $\mathcal{H}$ is the orthogonal projection from $\bigwedge^*(M)$ to $\mathcal{H}$ with respect to the inner product $(\ , \ )$.

  From the Hodge theorem, since $I = \mathcal{H} + \triangle \circ G$, we can write (called the Hodge-decomposition)

  Corollary(Hodge-decomposition)

$$
\begin{aligned}
\Lambda^r(M) &= \triangle(\Lambda^r(M)) \oplus \mathcal{H}^r(M) \\
&= d\delta\Lambda^r(M) \oplus \delta d\Lambda^r(M) \oplus \mathcal{H}^r(M) \\
&= d\Lambda^{r-1}(M) \oplus \delta\Lambda^{r+1}(M) \oplus \mathcal{H}^r(M).
\end{aligned}
$$

  To prove this theorem, basically we need to show tow things: (1): $\mathcal{H}$ **is a finite dimensional vector space**, (2): Write $\bigwedge^*(M) = \mathcal{H} \oplus \mathcal{H}^\perp$, where $\mathcal{H}^\perp$ is the orthogonal complement of $\mathcal{H}$ with respect to $(\ , \ )$, we need to show that $\triangle : \mathcal{H}^\perp \to \mathcal{H}^\perp$ **and** $\triangle$ **is one-to-one and onto.** (note that: for every $\phi \in \bigwedge^*(M), \psi \in \mathcal{H}$, $(\triangle\phi, \psi) = (\phi, \triangle\psi) = 0$, so $\triangle\phi \in \mathcal{H}^\perp$. Hence $\triangle : \mathcal{H}^\perp \to \mathcal{H}^\perp$). Once (1) and (2) are proved, then we take $G|_{\mathcal{H}} = 0$, and $G|_{\mathcal{H}^\perp} = \triangle^{-1}$. This will prove the Hodge theorem.

To do so, we first note that the operator $\triangle$ is positive (i.e. its eigenvalues are all positive). In fact, write $P = d + \delta$. Then it is easy to verify that both P are $\triangle$ are self-dual, and $\triangle = P^2$. Hence

$$(\triangle\phi, \phi) = (P\phi, P\phi) = (d\phi, d\phi) + (\delta\phi, \delta\phi) \geq 0.$$

So $\triangle$ is an elliptic self-adjoint operator. We therefore use the "theory of elliptic (self-adjoint) differential operator". To do so, we need first introduce the concept of "Sobolov space".

Let s be a nonnegative integer. Define the inner product $(\ ,\ )_s$ on $\bigwedge^*(M)$ as follows: for every $f_1, f_2 \in \bigwedge^*(M)$, define

$$(f_1, f_2)_s = \sum_{k=0}^{s} \int_M < \bigtriangledown^k f_1, \bigtriangledown^k f_2 > *1,$$

$$\|f_1\|_s^2 = (f_1, f_1)_s,$$

where $*1$ is the volume form on M. Let $H_s(M)$ be the completion of $\bigwedge^*(M)$ with respect to the Sobolov norm $\|\ \|_s$, which is called the 'Sobolov space.

We use the following three facts(proofs are omitted):

- **Garding's inequality**: *There exist constant $c_1, c_2 > 0$, such that for every $f \in \bigwedge^*(M)$, we have*

$$(\triangle f, f) \geq c_1\|f\|_1^2 - c_2\|f\|_0^2.$$

Remark: This is a variant of so-called *Bocher technique.*

To state the second fact, we introduce the concept of *weak derivative*: Write $P = d + \delta$ and $\triangle = P^2$. For $\phi \in H_s(M)$ and $\psi \in H_t(M)$, we say $P\phi = \psi$(weak), if for every test form $f \in \bigwedge^*(M)$, we have $(\phi, Pf) = (\psi, f)$. In similar way, $\triangle\phi = \psi$(weak) is defined. If $\phi \in H_s(M)$, $\psi \in H_t(M)$, and $P\phi = \psi$(weak), we denote it by $P\phi \in H_t(M)$.

- **Regularity of the operator** P: *If $\phi \in H_0(M)$ and $P\phi \in \bigwedge^*(M)$, then $\phi \in \bigwedge^*(M)$.*

- **Rellich Lemma**: *If $\{\phi_i\} \subset \bigwedge^*(M)$ is bounded in the $\| \ \|_1$, then it has a Cauchy subsequence with respect to the norm $\| \ \|_0$.*

  The above theorem about the **Regularity of the operator** P implies the following lemma

- **The weak form of the Wyle lemma**: *If $\phi \in H_1(M)$, and $\triangle\phi = \psi$(weak) with $\psi \in \bigwedge^*(M)$, then $\phi \in \bigwedge^*(M)$.*

Proof of the Hodge Theorem. We first prove that $\mathcal{H}$ is a finite dimensional vector space. If not, there exists an infinite orthonormal set $\{\omega_1, \ldots, \omega_n, \cdots\}$. By Garding's inequality, there exist constants c_1, c_2 such that for all i, we have

$$\|\omega_i\|_1^2 \le \frac{1}{c_1}\{(\triangle\omega_i, \omega_i) + c_2\|\omega_i\|_0^2\} = \frac{c_2}{c_1}.$$

By Rellich Lemma, $\{\omega_i\}$ must have a Cauchy subsequence with respect to the norm $\| \ \|_0$, which is impossible, since $\|\omega_i - \omega_j\|_0^2 = 2$ for $i \neq j$. This proves that $\mathcal{H}$ **is a finite dimensional vector space**.

Next, write

$$\bigwedge^*(M) = \mathcal{H} \oplus \mathcal{H}^\perp,$$

where $\mathcal{H}^\perp$ is the orthogonal complement of $\mathcal{H}$ with respect to $(\ , \ )$. We now prove a simpler version of Garding's inequality:

Garding's Lemma *there exists a positive constant c_0 such that for all $f \in \mathcal{H}^\perp$, we have*

$$\|f\|_1^2 \le c_0(\triangle f, f).$$

Proof. If not, there exists a sequence $f_i \in \mathcal{H}^\perp$ with $\|f_i\|_1 = 1$ and $(\triangle f_i, f_i) \to 0$. From Rellich lemma, we assume, WLOG, that f_i is convergent with respect to $\| \|_0$, i.e. there exists $F \in H_0(M)$ such that $\lim_{i\to+\infty} \|F - f_i\|_0 = 0$. We claim that $F = 0$. In fact, from above, $(\triangle f_i, f_i) = \|Pf_i\|_0^2 \to 0$, hence for every $\phi \in \bigwedge^*(M)$,

$$(F, P\phi) = \lim_{i\to+\infty}(f_i, P\phi) = \lim_{i\to+\infty}(Pf_i - \phi) = 0.$$

Hence $PF = 0$ (weak). From the regularity of P, we have $F \in \bigwedge^*(M)$. Hence

$$\triangle F = P(PF) = 0,$$

so $F \in \mathcal{H}$. Also, since $f_i \in \mathcal{H}^\perp$, we have, for every $\phi \in \mathcal{H}$,

$$(F, \phi) = \lim_{i\to+\infty}(f_i, \phi) = 0,$$

so $F \in \mathcal{H}^\perp$. Thus $F \in \mathcal{H} \cap \mathcal{H}^\perp$. This implies that $F = 0$. This means that $\lim_{i\to+\infty} \|f_i\|_0 = 0$. Now, by the Garding inequality, There exist constant $c_1, c_2 > 0$, such that

$$(\triangle f_i, f_i) \geq c_1\|f_i\|_1^2 - c_2\|f_i\|_0^2.$$

Because, from above, both $(\triangle f_i, f_i)$ and $\|f_i\|_0^2$ converge to zero, so $\lim_{i\to+\infty} \|f_i\|_1 = 0$, which contradicts the assumption that $\|f_i\|_1 = 1$. This proves Garding's lemma.

We now prove that $\triangle : \mathcal{H}^\perp \to \mathcal{H}^\perp$ *and* $\triangle$ *is one-to-one and onto.*

First we show that $\triangle : \mathcal{H}^\perp \subset \mathcal{H}^\perp$. In fact, for every $\phi \in \bigwedge^*(M), \psi \in \mathcal{H}$,

$$(\triangle\phi, \psi) = (\phi, \triangle\psi) = 0,$$

so $\triangle\phi \in \mathcal{H}^\perp$. To show $\triangle$ is one-to-one, let $\phi_1, \phi_2 \in \mathcal{H}^\perp$, and assume that $\triangle\phi_1 = \triangle\phi_2$. Then, from one hand, $\phi_1 - \phi_2 \in \mathcal{H}^\perp$. On the other hand, since $\triangle(\phi_1 - \phi_2) = 0$, $\phi_1 - \phi_2 \in \mathcal{H}$. Hence $\phi_1 = \phi_2$. It remains to show that $\triangle$ is onto. i.e. for every $f \in \mathcal{H}^\perp$, there exists $\phi \in \mathcal{H}^\perp$ such that $\triangle\phi = f$. This gets down to solve the differential equation $\triangle\phi = f$ (with unknown ϕ). Let B be the closure of $\mathcal{H}^\perp$ in $H_1(M)$. From Wyle's theorem, we only need to solve $\triangle\phi = f$ in the weak sense, i.e. there exists $\phi \in B$ such that, for every $g \in \bigwedge^*(M)$,

$$(\phi, \triangle g) = (f, g).$$

Since $\bigwedge^*(M) = \mathcal{H} \oplus \mathcal{H}^\perp$, we can write $g = g_1 + g_2$ where $g_1 \in \mathcal{H}, g_2 \in \mathcal{H}^\perp$. So the above identity is equivalent to every $g_2 \in \mathcal{H}^\perp$,

$$(\phi, \triangle g_2) = (f, g_2).$$

So the proof is reduced to the following statement: *for every $f \in \mathcal{H}^\perp$, there exists $\phi \in B$ such that, for every $g \in \mathcal{H}^\perp$,*

$$(\phi, \triangle g) = (f, g).$$

We now use the **Riesz representation** theorem to prove this statement. In fact, for every $\phi, \psi \in \mathcal{H}^\perp$, define $[\phi, \psi] = (\phi, \triangle \psi)$, and consider the linear transformation $L : B \to \mathbf{R}$ defined by $l(g) = (f, g)$ for every $g \in B$. Our goal is to show that we can extend $[\ ,\ ]$ to B such that l is continuous with respect to $[\ ,\ ]$ (or bounded). Then by **Riesz representation** theorem, there exists $\phi \in B$ such that, for every $g \in B$ (in particular for $g \in \mathcal{H}^\perp$),

$$l(g) = [\phi, g].$$

This will prove our statement. To extend $[\ ,\ ]$, we compare $[\ ,\ ]$ with $(\ ,\ )_1$. From definition, $[\ ,\ ]$ is bilinar. From Garding's inequality, for every $\phi \in \mathcal{H}^\perp$,

$$[\phi, \phi] = (\phi, \triangle \phi) \geq \frac{1}{c_0} \|\phi\|_1^2.$$

On the other hand,

$$[\phi, \phi] = (\phi, \triangle \phi) = \|P\phi\|_0.$$

By direct verification, we have, for every $\phi \in \bigwedge^*(M)$,

$$\|P\phi\|_0^2 \leq c\|\phi\|_1^2.$$

Hence

$$[\phi, \phi] \leq c\|\phi\|_1^2.$$

So $[\ ,\ ]$ and $(\ ,\ )_1$ are equivalent on $\mathcal{H}^\perp$. So there exists an unique continuation on B, and for every $g \in B$, we have

$$[g, g] \geq \frac{1}{c_0} \|g\|_1^2.$$

To show that l is continuous with respect to $[\ ,\ ]$(or bounded), we notice that

$$|l(g)| = |(f,g)| \leq \|f\|_0\|g\|_0 \leq \|f\|_0\|g\|_1 \leq \sqrt{c_0}\|f\|_0\sqrt{[g,g]}.$$

So the claim is proved. This finishes the proof that $\triangle$ is onto.

To prove Hodge's theorem, since, from above, $\triangle : \mathcal{H}^\perp \to \mathcal{H}^\perp$ is one-to-one and onto, we let $G : \bigwedge^*(M) \to \bigwedge^*(M)$ be defined as follows: $G|_{\mathcal{H}} = 0$, and $G|_{\mathcal{H}^\perp} = \triangle^{-1}$. Then we see that ker$G = \mathcal{H}$ and $I = \mathcal{H} + \triangle \circ G$. The rest of properties are also easy to verify.

This finishes the proof.

- **Application of the Hodge Theory**. Let M be a compact manifold. Denote by $\Lambda^r(M)$ the set of all r-forms on M. Clearly $\Lambda^0(M)$ is the set of all differential functions on M. By the rule of the exterior multiplication, we see that $0 \leq r \leq n$.

  The *exterior differential* operator is a map $d : \Lambda^r(M) \to \Lambda^{r+1}(M)$, which satisfies conditions:

  (i) d is $\mathbf{R}$-linear;

  (ii) For $f \in \Lambda^0(M)$, df is the usual differential of f, and $d(df) = 0$;

  (iii) $d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^r \phi \wedge d\psi$ for any $\phi \in \Lambda^r(M)$ and any ψ.

  There are three important properties for d: (a) $d^2 = 0$ (called the Poincare lemme), (b) For $\omega \in \Lambda^1(M)$ and $X, Y \in \Gamma(TM)$, we have

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]).$$

  (c) If $F : M \to N$, then $F^* \circ d = d \circ F^*$.

  A differential r-form $\phi \in \Lambda^r(M)$ is said to be *closed* if $d\theta = 0$, and $\phi \in \Lambda^r(M)$ is said to be *exact* if there exists $\eta \in \Lambda^{r-1}(M)$ such that $\phi = d\eta$. Since $d \circ d = 0$, we know that every exact form is also closed. Let $Z^r(M, \mathbf{R})$ denote the set of all (smooth) closed $r$forms on

M, and let $B^r(M,\mathbf{R})$ denote the set of all (smooth) exact $r$forms on M. Then $B^r(M,\mathbf{R}) \subset Z^r(M,\mathbf{R})$ which allows us to form the quotient space $H^r(M,\mathbf{R}) := Z^r(M,\mathbf{R})/B^r(M,\mathbf{R})$, called the *deRham cohomology group* of dimension r. Set

$$H^*(M,\mathbf{R}) = H^0(M,\mathbf{R}) \oplus H^1(M,\mathbf{R}) \oplus \cdots \oplus H^m(M,\mathbf{R}),$$

which is an algebra with the exterior multiplication.

Theorem (the deRham Theorem) *There is a natural isomorphism of $H^*(M,\mathbf{R})$ and the cohomology ring of M.*

As an application of Hodge theory, we can study $H^r(M,\mathbf{R})$ using the nice representation of harmonic forms as follows

Theorem(Representing Cohomology Classes by Harmonic Fomrs). *Each deRham cohomology class on (M,g) contains a unique harmonic representative.*

Proof. Let $h : \Lambda^r(M) \to \mathcal{H}^r(M)$ be the orthogonal projection. If $\omega \in \Lambda^r(M)$ is closed, then according to the Hodge decomposition, we have

$$\omega = d\alpha + h(\omega)$$

which implies that $[\omega] = [h(\omega)] \in H^r(M,\mathbf{R})$. Since $\mathcal{H}^r(M) \perp d\Lambda^{r-1}(M)$ we see that two different harmonic forms must belong to two different deRham cohomology classes. In fact, if $\gamma_1, \gamma_2 \in \mathcal{H}^r(M)$ and $[\gamma_1] = [\gamma_2]$, then $\gamma_1 - \gamma_2 = d\alpha$. But, $d\alpha \perp (\gamma_1 - \gamma_2)$, thus $d\alpha = 0$, so $\gamma_1 = \gamma_2$. Hence $h(\omega)$ is unique in $H^r(M,\mathbf{R})$.

From the proof of the Hodge theorem, **we see that** $\dim \mathcal{H}^r(M) < +\infty$ **if** M **is finite, so we get that** $\dim H^r(M,\mathbf{R}) < +\infty$ **if** M **is compact**.

Let M be a compact, oriented, differentiable manifold of dimension m. We define a bilinear function

$$H^r(M,\mathbf{R}) \times H^{m-r}(M,\mathbf{R}) \to \mathbf{R}$$

by sending

$$([\phi], [\psi]) \mapsto \int_M \phi \wedge \psi.$$

Observe that the bilinear map is well-defined, i.e. if $\phi_1 =$ $phi + d\xi$, then, by Stoke's theorem,

$$\int_M \phi_1 \wedge \psi = \int_M \phi \wedge \psi.$$

Theorem. *Poincare duality theorem. The bilinear function above is non-singular pairing and consequently determines isomorphisms of $\mathcal{H}^{m-r}(M)$ with the dual space of $\mathcal{H}^r(M)$:*

$$H^{m-r}(M, \mathbf{R}) \simeq (H^r(M, \mathbf{R}))^*.$$

In fact, given a non-zero cohomology class $[\phi] \in H^r(M, \mathbf{R})$, we must find a non-zero cohomology class $[\psi] \in H^{m-r}(M, \mathbf{R})$, such that $([\phi], [\psi]) \neq 0$. Choose a Riemannian structure. We can assume that ϕ is harmonic, and $\phi \neq$. Since $*\triangle = \triangle *$, we have that $*\phi$ is also harmonic, and $*\phi \in H^{m-r}(M, \mathbf{R})$. Now,

$$([\phi], [\psi]) = \int_M \phi \wedge *\phi = \|\phi\|^2 \neq 0.$$

So the statement is proved.

The r-th *Betti number* $\beta_r(M)$ of (M, g) is defined by

$$\beta_r(M) = \dim H^r(M, R) = \dim \mathcal{H}^r.$$

Then we have

$$\beta_r(M) = \beta_{m-r}(M).$$

The *Euler-Poincare* characteristic number $\chi(M)$ of (M, g) is defined by

$$\chi(M) = \sum_{r=0}^{m} (-1)^r \dim H^r(M, R) = \sum_{r=0}^{m} (-1)^r \beta_r(M).$$

Then, we have the statement that *if* $m = \dim M$ *is odd, then* $\chi(M) = 0$..

Another statement we can prove(will be proved later) is *Let* (M, g) *be a compact oriented Riemannian manifold without boundary. If its Ricci curvature is positive, then*

$$\beta_1(M) = \beta_{m-1}(M) = 0.$$