Math 6397 Riemannian Geometry II, HW#3 (due on Monday, March 20)

Differential Operators on Riemannian Manifolds

Remark: This set of homework is about the concept of exterior derivative, volume form, interior product, divergence operator, gradient, Laplace operator, Hodge star operator, etc. This problem set may be very hard to you. Try your best!

1. Let \((M, g)\) be a Riemannian manifold with the Levi-Civita connection \(D\). Let \(\{e_i\}\) be a local frame field on \(M\) and \(\{\omega^i\}\) be its dual, i.e. \(\omega^j(e_i) = \delta^j_i\). Show that, for every smooth differential \(r\)-form \(\theta\),

\[
d\theta = \sum_i \omega^i \wedge D_{e_i} \theta.
\]

2. Let \(M\) be a Riemannian manifold with the Levi-Civita connection \(D\). Let \(X\) be a smooth tangent vector field on \(M\). Define

\[
div(X) = tr\{Y \to DYX\}.
\]

Show that, in local coordinate \((U; x^i)\),

\[
div(X) = \frac{1}{\sqrt{G}} \sum_{i=1}^m \frac{\partial}{\partial x^i}(\sqrt{G}X^i),
\]

where \(G = det(g_{ij}), g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j), X^i = dx^i(X)\).

3. Let \((M, g)\) be a Riemannian manifold with the Levi-Civita connection \(D\). Let \(f \in C^\infty(M)\), define a tangent vector field \(\nabla f\) on \(M\), by

\[
g(\nabla f, X) = df(X) = X(f),
\]

for every smooth tangent vector field \(X\). The tangent vector field \(\nabla f\) is called the gradient of \(f\). Show that, in local coordinate \((U; x^i)\),

\[
\nabla f = \sum_{j=1}^m \left( \sum_{i=1}^m g^{ij} \frac{\partial f}{\partial x^i} \right) \frac{\partial}{\partial x^j},
\]

where \(G = det(g_{ij}), g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j), (g^{ij}) = (g_{ij})^{-1}\).
4. Let \((M, g)\) be a Riemannian manifold with the Levi-Civita connection \(D\). Let \(f \in C^\infty(M)\), define \(\triangle f = \text{div}(\nabla f)\). It is called the Laplace operator. Show that, in local coordinate \((U; x^i)\),

\[
\triangle f = \frac{1}{\sqrt{G}} \sum_{i=1}^{m} \frac{\partial}{\partial x^i} \left( \sum_{j=1}^{m} \sqrt{G}g^{ij} \frac{\partial f}{\partial x^j} \right),
\]

where \(G = \det(g_{ij}), g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j), (g^{ij}) = (g_{ij})^{-1}\).

5. Let \((M, g)\) be a Riemannian manifold with the Levi-Civita connection \(D\). Let \(\eta\) be its volume form of \(M\). Prove that, for every smooth tangent vector field \(X\),

\[
d(i(X)\eta) = \text{div}(X)\eta,
\]

where \(i(X)\) is the interior product, i.e.

\[(i(X)\eta)(X_1, \ldots, X_{m-1}) = \eta(X, X_1, \ldots, X_{m-1}),\]

for every tangent vector fields \(X_1, \ldots, X_{m-1}\).

6. Let \((M, g)\) be an oriented Riemannian manifold with the Levi-Civita connection \(D\). Let \(\{e_i\}\) be a local frame field on \(M\) compatible with the orientation of \(M\). Let \(g_{ij} = g(e_i, e_j)\), and \((g^{ij}) = (g_{ij})^{-1}\). Prove that the codifferential operator \(\delta\) can be written as

\[
\delta \alpha = - \sum_{i,j=1}^{m} g^{ij}(De_i\alpha), \quad \text{for every } \alpha \in \Lambda^{r+1}(M),
\]

i.e., for every tangent vector fields \(X_1, \ldots, X_r\).

\[
\delta \alpha(X_1, \ldots, X_r) = - \sum_{i,j=1}^{m} g^{ij}(De_i\alpha)(e_j, X_1, \ldots, X_r).
\]

7. Prove that \(\text{div}(X) = -\delta(\alpha_X)\), where \(\alpha_X\) is the 1-form defined by \(\alpha_X(Y) = g(X, Y)\) for every smooth tangent vector field \(Y\). Prove that, for every smooth tangent vector field \(X\), \(\text{div}(X) = -\delta(\alpha_X)\), where \(\delta\) is the codifferential operator.
8. Let $(M, g)$ be an oriented Riemannian manifold with the Levi-Civita connection $D$. Denote by $\triangle$ the Hodge-Laplace operator. Let $f \in C^\infty(M)$. Prove that $\triangle f = -\text{div}(\nabla f)$.

9. (The purpose is to find an expression of the Hodge-star operator under the local coordinates (not necessarily orthonormal)). Let $(M, g)$ be an oriented Riemannian manifold with the Levi-Civita connection $D$. Let $(U, x^1, \ldots, x^m)$ be a local coordinate. Let $*$ be the Hodge operator. Show that, for every

$$\omega = \frac{1}{r!} \sum_{i_1, \ldots, i_r} a_{i_1 \ldots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r},$$

$$*\omega = \frac{\sqrt{G}}{r!(n-r)!} \delta_{i_1 \ldots i_m}^{i_{r+1} \ldots i_m} a_{i_1 \ldots i_r} dx^{i_{r+1}} \wedge \cdots \wedge dx^{i_m},$$

where

$$a_{i_1 \ldots i_r} = \sum_{j_1 \ldots j_r} g^{ij_1} \cdots g^{jr} a_{i_1 \ldots i_r},$$

here $g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j)$, $G = \det(g_{ij})$, $(g^{ij}) = (g_{ij})^{-1}$ and $\delta_{i_1 \ldots i_m}^{i_{r+1} \ldots i_m}$ is the Levi-Civita permutation symbol, i.e. $\delta_{i_1 \ldots i_m}^{i_{r+1} \ldots i_m} = 1$ if $(i_1 \cdots i_m)$ is an even permutation of $(12 \ldots m)$, $\delta_{i_1 \ldots i_m}^{i_{r+1} \ldots i_m} = -1$ if $(i_1 \cdots i_m)$ is an odd permutation of $(12 \ldots m)$, $\delta_{i_1 \ldots i_m}^{i_{r+1} \ldots i_m} = 0$ otherwise.