Math 6397 Riemannian Geometry II
Sheaf Cohomology
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0.1 Sheaves

Definition 0.1. *A* **sheaf** $\mathcal{F}$ *over a complex manifold* X *consists of, for each open set* $U \subset X$*, an abelian group (or vector spaces, rings, or any desired object)* $\mathcal{F}(U)$ *(also denoted* $\Gamma(\mathcal{F}, U)$ *and called the set of sections over* U*), and a collection of restriction maps such that for each* $U \subset V \subset X$, $\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$*, and satisfy:*

(1) *Identity:* $\rho_{U,U} = id|_{\mathcal{F}(U)}$*,*

(2) *Compatibility: If* $U \subset V \subset W \subset X$*, then* $\rho_{V,U} \circ \rho_{W,V} = \rho_{W,U}$*;*

(3) *Sheaf axiom (gluing): Let* $U = \cup_\alpha U_\alpha$ *and* $\sigma_\alpha|_{U_\alpha \cap U_\beta} = \sigma_\beta|_{U_\alpha \cap U_\beta}$ *for all* α, β*, then there exists a (unique)* $\sigma \in \mathcal{F}(U)$ *such that* $\sigma_\alpha = \sigma|_\alpha$ *for all* α*.*

If only (1) and (2) are satisfied, then $\mathcal{F}$ is call a **presheaf**.

Every presheaf $S = \{S(U), \rho_{UV}\}$ over X determines, in a natural way, a sheaf $\mathcal{F}$ which is defined as follows: For every $x \in X$ the subsystem $\{S(U), \rho_{UV}, U \in \mathcal{U}_X\}$ is directed with respect to inclusion of open neighborhoods of x. Thus the direct limit $\mathcal{F}_x := \varinjlim_{x \in U} S(U)$ and the maps $\rho_{U,x} : S(U) \longrightarrow \mathcal{F}_x$ are defined. We let $\mathcal{F} := \bigcup_{x \in X} \mathcal{F}_x$ and define $\pi : \mathcal{F} \longrightarrow X$ by $\pi(\mathbf{f}) = x$, as $\mathbf{f} \in \mathcal{F}_x$. Every element $f \in S(U)$ determines the set $D_f := \bigcup_{x \in U} \rho_{U,x}(f) \subset \mathcal{F}$. The system of subsets of $\mathcal{F}$, $\{D_f | U \in \mathcal{U}_X,\ f \in S(U)\}$, is a basis for a topology on $\mathcal{F}$. We endow $\mathcal{F}$ with this topology. Then it is easy to verify that $(\mathcal{F}, \pi, X)$ is a sheaf over X.

Let's us consider a few familiar examples, each with the natural sections and restriction maps.

Example 1. $\mathcal{O}_X$*, the sheaf of holomorphic functions on* X*.*

Example 2. Ω_X^p*, the sheaf of holomorphic* p*-forms on* X*.*

Example 3. $\mathcal{A}_X^n$*, the sheaf of* n*-forms on* X*.*

Example 4. $\mathcal{A}_X^{a,b}$*, the sheaf of* (a,b)*-forms on* X*.*

Example 5. The *skyscraper* sheaf $\mathbf{C}_p$ given by $\mathbf{C}_p(U) = \mathbf{C}$ if $p \in U$, and $\mathbf{C}_p(U) = 0$ if $p \notin U$ along with the natural restriction maps.

0.2 Cech cohomology

Let $\mathcal{F}$ be an abelian group sheaf over a complex manifold X, U be an open set of X, $\forall s_1,\ s_2 \in \Gamma(U, \mathcal{F})$, we can define the section $s_1+s_2 \in \Gamma(U, \mathcal{F})$ as following

$$(s_1+s_2)(x) = s_1(x) + s_2(x), \qquad \forall x \in U.$$

Since the group operation is continuous in the topology of $\mathcal{F}$, so s_1+s_2 is continuous, and $\pi(s_1+s_2) = id_U$ is obviously, therefore s_1+s_2 is still a section over U. After endowing with such addition operation, $\Gamma(U, \mathcal{F})$ is an abelian group.

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of topological space X. For $\forall$ nonnegative integer p,

$$\begin{aligned} f : \{|\sigma| = U_{i_0} \cap \cdots \cap U_{i_p} \neq \emptyset\} &\longrightarrow \{\Gamma(|\sigma|, \mathcal{F})\} \\ U_{i_0} \cap \cdots \cap U_{i_p} &\to f_{i_o \cdots i_p} \end{aligned}.$$

If $U_{i_0} \cap \cdots \cap U_{i_p} \emptyset$, we take $f_{i_o \cdots i_p} = 0$. Such $f = \{f_{i_0 \cdots i_p}\}$ is called a **p-cochain** of $\mathcal{U}$ with coefficients in the sheaf $\mathcal{F}$. We use $C^p(\mathcal{U}, \mathcal{F})$ to denote the set of all p-cochains of $\mathcal{U}$ with coefficients in the sheaf $\mathcal{F}$. For $\forall \{f_{i_0 \cdots i_p}\},\ \{g_{i_0 \cdots i_p}\} \in C^p(\mathcal{U}, \mathcal{F})$, defining the addition operation

$$\{f_{i_0 \cdots i_p}\} + \{g_{i_0 \cdots i_p}\} = \{f_{i_0 \cdots i_p} + g_{i_0 \cdots i_p}\}$$

then $C^p(\mathcal{U}, \mathcal{F})$ becomes an abelian group, we called $C^p(\mathcal{U}, \mathcal{F})$ p-dimensional **cochains group** of $\mathcal{U}$ with **coefficients in sheaf** $\mathcal{F}$.

Now we define the operator

$$\begin{aligned} \delta_p : C^p(\mathcal{U}, \mathcal{F}) &\longrightarrow C^{p+1}(\mathcal{U}, \mathcal{F}) \\ f &\to \delta_p f \end{aligned}$$

where

$$(\delta_p f)_{i_0 \cdots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k f_{i_0 \cdots \widehat{i_k} \cdots i_{p+1}}. \tag{1}$$

In the right hand side of (1), each $f_{i_0 \cdots \widehat{i_k} \cdots i_{p+1}}$ restricts to $U_{i_0} \cap \cdots \cap U_{i_{p+1}}$ and proceeds the addition operation in $\Gamma(U_{i_0} \cap \cdots \cap U_{i_{p+1}}, \mathcal{F})$. It is easy to verify δ_p is a homeomorphism of group, and $\delta_{p+1} \circ \delta_p = 0;\ p \geq 1$. $Z^p(\mathcal{U}, \mathcal{F}) := Ker\ \delta_p \subset C^p(\mathcal{U}, \mathcal{F}),\ p \geq 0$, is called the p-dimensional **cocycles group** of $\mathcal{U}$ with **coefficients in sheaf** $\mathcal{F}$, and $B^p(\mathcal{U}, \mathcal{F}) = Im\ \delta_{p-1},\ p \geq 1$, is called the p-dimensional **coboundaries group** of $\mathcal{U}$ with **coefficients in sheaf** $\mathcal{F}$, and $B^0(\mathcal{U}, \mathcal{F}) \equiv 0$. From $\delta_{p+1} \circ \delta_p \equiv 0,\ B^p(\mathcal{U}, \mathcal{F}) \subset Z^p(\mathcal{U}, \mathcal{F})$.

Definition 0.2.

$$H^p(\mathcal{U}, \mathcal{F}) = \begin{cases} Z^p(\mathcal{U}, \mathcal{F}) \big/ B^p(\mathcal{U}, \mathcal{F}), & p \geq 1 \\ Z^0(\mathcal{U}, \mathcal{F}), & p = 0 \end{cases}, (2)$$

$H^p(\mathcal{U}, \mathcal{F})$ is called the p-dimensional **cohomology group** *or more precisely the p-dimensional cohomology gy group of $\mathcal{U}$ with coefficients in the sheaf $\mathcal{F}$.*

The $H^p(\mathcal{U},\mathcal{F})$ defined above is dependent of the open covering $\mathcal{U}$ of X. Below, we shall define the cohomology group is only dependent of X, but not of the covering. In general, there are two method to treat this, one is the Čech's method, another is the Grotherdisk's method.

Čech's method is to introduce a partial order in the set of all the open coverings. Let $\mathcal{W}$ and $\mathcal{U}$ be two coverings of X, we say $\mathcal{W}$ is a refine covering of $\mathcal{U}$, if there exists a mapping

$$\begin{aligned} \rho : \mathcal{W} &\longrightarrow \mathcal{U} \\ W &\rightarrow \rho(W) \end{aligned}$$

such that at $\rho(W) \supset W$, and use $\mathcal{W} \prec \mathcal{U}$ to denote that $\mathcal{W}$ is a refinement of $\mathcal{U}$. ρ is called a refining mapping from $\mathcal{W}$ to $\mathcal{U}$. ρ induces a mapping, which we still use the notation ρ, from $C^p(\mathcal{U},\mathcal{F}) \longrightarrow C^p(\mathcal{W},\mathcal{F})$; $p = 0,1,2,\cdots$,

$$\rho : C^p(\mathcal{U},\mathcal{F}) \longrightarrow C^p(\mathcal{W},\mathcal{F}) \tag{3}$$

for $\forall f \in C^p(\mathcal{U},\mathcal{F})$, $\forall|\sigma| = W_{i_0} \cap \cdots \cap W_{i_p} \neq \emptyset$

$$(\rho f)_{i_0\cdots i_p} = \rho f(W_{i_0},\cdots,W_{i_p}) = r_{|\sigma|} f\left(\rho(W_{i_0}),\cdots,\rho(W_{i_p})\right).$$

It is easy to verify that (3) is a group homeomorphism and ρ is commutative with all $\{\delta_p\}$, i.e., $\rho\delta_p = \delta_p\rho$; $p = 0,1,2,\cdots$, therefore ρ restricts to $Z^p(\mathcal{U},\mathcal{F})$, $B^p(\mathcal{U},\mathcal{F})$ which are both the group homeomorphisms from $Z^p(\mathcal{U},\mathcal{F})$, $B^p(\mathcal{U},\mathcal{F})$ to $Z^p(\mathcal{U},\mathcal{F})$, $B^p(\mathcal{U},\mathcal{F})$ respectively, furthermore ρ induces a group homeomorphism.

$$\rho_* : H^p(\mathcal{U},\mathcal{F}) \longrightarrow H^p(\mathcal{U},\mathcal{F}).$$

We have to point out, if $\mathcal{W} \prec \mathcal{U}$, there exists a refining mapping $\rho : \mathcal{W} \longrightarrow \mathcal{U}$, but in general the refining mapping from $\mathcal{W}$ to $\mathcal{U}$ is not unique, so that whose inducing mapping from $C^p(\mathcal{U},\mathcal{F})$ to $C^p(\mathcal{W},\mathcal{F})$ is also different, but we can prove that the homomorphism $\rho_* : H^p(\mathcal{U},\mathcal{F}) \longrightarrow H^p(\mathcal{W},\mathcal{F})$ in (2) is the same as different refining mappings from $\mathcal{W}$ to $\mathcal{U}$, i.e., ρ_* only depends on $\mathcal{W}$ and $\mathcal{U}$, but not on the particular refining mapping ρ. The set of all open covering of X becomes a directed set for the refinement relation $\prec$, as X is Hausdorff and paracompact topological space, the direct limit of $H^p(\mathcal{U},\mathcal{F})$

$$\check{H}^p(X,\mathcal{F}) = \varinjlim_{\mathcal{U}} H^p(\mathcal{U},\mathcal{F})$$

defines a cohomology group, which only depends on X. We call $\check{H}^p(X,\mathcal{F})$ the p-dimensional Čech cohomology group wit h coefficients in sheaf $\mathcal{F}$.

Theorem 0.3. *Let $\mathbf{C}_p$ be the skyscraper sheaf. Then (i) $H^0(M,\mathbf{C}_p) = \mathbf{C}$, (ii) $H^1(M,_p) = 0$.*

The assertion of (i) is trivial. As for (ii), consider a cohomology class $\xi \in H^1(M,\mathbf{C}_p)$, which is represented by a cocycle in $Z(\mathcal{U},\mathbf{C}_p)$. The covering $\mathcal{U}$ has a refinement $\mathcal{B} = \{V_\alpha\}$ such that the point p is contained in only one V_α. But then $Z(\mathcal{U},\mathbf{C}_p) = 0$ and hence $\xi = 0$. This finishes the proof.

0.3 Sheaf Morphisms, Long Exact Sequences in Cohomology

Definition 0.4. *Let X be a complex manifold and let $\mathcal{F}$, $\mathcal{G}$ be two sheaves over X. A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ consists of morphisms $\varphi_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ such that the φ_U commutes with the restriction maps for $\mathcal{F}$ and $\mathcal{G}$, i.e. for $\forall V,\ U \in \mathcal{U}_X$ and $V \subset U$, then*

$$\varphi_V \circ \rho^1_{UV} = \rho^2_{UV} \circ \varphi_U.$$

If there exists another presheaf homomorphism $\psi = \{\psi_U\}_{U \in \mathcal{U}_X} : \mathcal{G} \longrightarrow \mathcal{F}$, such that, for every $U \in \mathcal{U}_X$,

$$\psi_U \circ \varphi_U = id_{\mathcal{F}(U)}$$

and

$$\varphi_U \circ \psi_U = id_{\mathcal{G}(U)}.$$

We call $\varphi = \{\varphi_U\}_{U \in \mathcal{U}_X}$ $\left(\psi = \{\psi_U\}_{U \in \mathcal{U}_X}\right)$ the **presheaf isomorphism***.*

Example. Let $\mathcal{O}^*_X$ be the sheaf of nonwhere vanishing holomorphic functions on X. It is a subsheaf of $\mathcal{O}_X$. Then

$$\exp : f \mapsto e^{2\pi i f}, \mathcal{O}_X \to \mathcal{O}^*_X$$

is a sheaf morphism.

Given a morphism $\varphi : \mathcal{F} \to \mathcal{G}$, we define ker φ and im φ. For ker φ, the naive definition works and define a subsheaf ker $\varphi \subset \mathcal{F}$:

$$(ker\ \varphi)(U) := ker\{\varphi_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)\}.$$

This construction in fact satisfies the sheaf axioms. However, the naive definition for the image would not, in general, be a sheaf. It is only a presheaf. For example, let X be a compact complex manifold, and consider the natural map

$$\varphi : \mathcal{O}_X \to \mathbf{C}_p \oplus \mathbf{C}_q.$$

The solution, to define the image is to *sheafify* the presheaf. This leads to the following definition

$$(Im\ \varphi)(U) := \{s \in \mathcal{G}(U) : \forall p \in U, \text{ there is } V \subset U, \text{ s.t. } p \in V, \text{ and } s|_V \in im(\varphi_V)\}.$$

The meaning of exact sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{\lambda} \mathcal{G} \xrightarrow{\mu} \mathcal{H} \longrightarrow 0$$

is that λ is injective, and μ is surjective, i.e. $\text{im}(\mu) = \mathcal{H}$.

Proposition 0.5. *Let*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\lambda} \mathcal{G} \xrightarrow{\mu} \mathcal{H} \longrightarrow 0$$

be an exact sequence of sheaves. Then for $\forall U \in \mathcal{U}_X$*,*

$$0 \longrightarrow \Gamma(U,\mathcal{F}) \xrightarrow{\lambda_U} \Gamma(U,\mathcal{G}) \xrightarrow{\mu_U} \Gamma(U,\mathcal{H})$$

is an exact sequence of section groups.

Proof. $Ker(\lambda_U) = 0$, since $\forall f \in \Gamma(U,\mathcal{F})$, $\lambda_U(f) = 0$, i.e., for $\forall x \in U$, $\lambda\big(f(x)\big) = 0$, since λ is injective, $f(x) = 0$. $\forall x \in U$, $f \equiv 0$, therefore (4.35) is exact at $\Gamma(U,\mathcal{F})$. Since $\mu \circ \lambda = 0$, $\mu_U \circ \lambda_U = 0$ by the definition of μ_U and λ_U, therefore $Im(\lambda_U) \subset Ker(\mu_U)$. For $\forall g \in \Gamma(U,\mathcal{G})$, if $\mu_U(g) = 0$, that is $\mu\big(g(x)\big) = 0$, for $\forall x \in U$. By the exactness of (4.34), $g(x) \in Im(\lambda)$, $\forall x \in U$, i.e., $Im\ g \subset Im(\lambda)$, hence there exists $f \in \Gamma(U,\mathcal{F})$ such that $\lambda_U(f) = g$. ☐

In general, the μ_U is not necessarily surjective. We provide an example to elucidate the fact.

Example: $X = \Delta^* = \{z \in \mathbf{C}^1 | 0 < |z| < 1\}$ is the punctured unit disc in $\mathbf{C}^1$, $\mathcal{O}$ is the sheaf of germs of holom orphic functions, $\mathcal{O}^*$ is the sheaf of germs of holomorphic functions without the zero, Z is the sheaf of germs of integral numbers, then we have following exact sequence of sheaves

$$0 \longrightarrow Z \xrightarrow{i} \mathcal{O} \xrightarrow{e} \mathcal{O}^* \longrightarrow 0 \tag{4}$$

where i is inclusion homomorphism, $e(\mathbf{f}_x) = (exp\ 2\pi i f)_x$, where $\mathbf{f}_x$ is the germ of f at x and f is a holomorphic on a neighborhood of x, $(exp\ 2\pi i f)_x$ in the germ of $exp\ 2\pi i f$ at x. It is easy to verify (4) is an exact sequence of sheaves. Now we consider the following sequence of group homomorphisms,

$$0 \longrightarrow \Gamma(\Delta^*, Z) \xrightarrow{i_{\Delta^*}} \Gamma(\Delta^*, \mathcal{O}) \xrightarrow{e_{\Delta^*}} \Gamma(\Delta^*, \mathcal{O}^*) \longrightarrow 0.$$

For the holomorphic function $z \in \Gamma(\Delta^*, \mathcal{O}^*)$, there is no $g \in \Gamma(\Delta^*, \mathcal{O})$ such that $exp(2\pi i g) = z$. In fact, the only solution is $g = \frac{1}{2\pi i} log\ z$, but $\frac{1}{2\pi i} log\ z$ is not the unique valued holomorphic functions on Δ^*.

Theorem 0.6. *Let*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow 0 \tag{4.52}$$

be an exact sequence over the topological space X*. Then we have the following exact sequence of the cohomology group*

$$\begin{aligned}
0 &\longrightarrow H^0(X,\mathcal{F}) \xrightarrow{\alpha_*} H^0(X,\mathcal{G}) \xrightarrow{\beta_*} H^0(X,\mathcal{H}) \\
&\xrightarrow{\delta_*} H^1(X,\mathcal{F}) \longrightarrow \cdots \qquad \cdots \\
&\cdots \qquad \cdots \qquad \cdots \qquad \cdots \\
&\xrightarrow{\delta_*} H^{k-1}(X,\mathcal{F}) \xrightarrow{\alpha_*} H^{k-1}(X,\mathcal{G}) \xrightarrow{\beta_*} H^{k-1}(X,\mathcal{H}) \\
&\xrightarrow{\delta_*} H^k(X,\mathcal{F}) \xrightarrow{\alpha_*} H^k(X,\mathcal{G}) \xrightarrow{\beta_*} H^k(X,\mathcal{H}) \\
&\xrightarrow{\delta_*} H^{k+1}(X,\mathcal{F}) \longrightarrow \cdots
\end{aligned}$$

where α_*, β_* *are the homomorphisms of cohomology group induced by sheaf homomorphisms* α, β *respectively, and the definition o f the connecting homomorphism* δ_* *will be given in the proof.*

0.4 Forms and Cohomology

Let M be a complex manifold of dimension n (so it is a real differentiable manifold of dimension $2n$). Suppose $M = \bigcup\limits_{\alpha\in I} U_\alpha$, where $\{U_\alpha\}_{\alpha\in I}$ is an open covering of M which consists of the local coordinate neighborhood of M. For $\forall p \in M$, let T_pM be the tangent space of M at point p. Then $TM = \bigcup\limits_{p\in M} T_pM$ is the tangent bundle of M, where $\pi : TM \longrightarrow M$ is defined by $\pi(T_pM) = p$. Let $\{U_\alpha\}_{\alpha\in I}$ be a coordinate system of M, i.e. each U_α, $\alpha \in I$, is the local coordinate neighborhood. Let $\pi^{-1}(U_\alpha) = \bigcup\limits_{p\in U_\alpha} T_p(M)$ and $\phi_\alpha : \pi^{-1}(U_\alpha) \overset{\sim}{\longrightarrow} U_\alpha \times R^n$. If $(x^1, \cdots, x^n, y^1, \cdots, y^n)$ is a local coordinate system of U_α, then, for $\forall p \in U_\alpha$, $T_p(M) = \{\sum\limits_{i=1}^{n} a^i \frac{\partial}{\partial x^i}|_p + \sum\limits_{i=1}^{n} b^i \frac{\partial}{\partial y^i}|_p \mid a^i, b^i \in R, i = 1, \cdots, n\}$. The *complexified tangent space* is

$$T_{\mathbf{C},p}(M) =: \mathbf{C}\otimes \mathrm{T_p(M)} = \left\{ \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{a^i} \frac{\partial}{\partial \mathrm{x^i}}|_{\mathrm{p}} + \sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{b^i} \frac{\partial}{\partial \mathrm{y^i}}|_{\mathrm{p}} \mid \mathrm{a^i}, \mathrm{b^i} \in \mathbf{C} \right\} = \mathbf{C}\left\{ \frac{\partial}{\partial \mathrm{x^i}}, \frac{\partial}{\partial \mathrm{y^i}} \right\}.$$

Define

$$\frac{\partial}{\partial z^i} = \frac{1}{2}(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i}) \quad \textit{and} \quad \frac{\partial}{\partial \bar{z}^i} = \frac{1}{2}(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y^i}); \quad 1 \le i \le n,$$

then we can write

$$T_{\mathbf{C},p}(M) = \mathbf{C}\left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j} \right\} = \mathbf{C}\left\{ \frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^i} \right\}.$$

Define a splitting of the complexified tangent space into two pieces:

$$T_{\mathbf{C},p}(M) = T_p^{1,0}(M) \oplus T_p^{0,1}(M),$$

where $T_p^{1,0}(M)$ is the *holomorphic tangent space* and $T_p^{0,1}(M)$ is the *antiholomorphic tangent space*, given by

$$T_p^{1,0}(M) = \mathbf{C}\left\{ \frac{\partial}{\partial z^i} \right\},$$

$$T_p^{0,1}(M) = \mathbf{C}\left\{ \frac{\partial}{\partial \bar{z}^i} \right\},$$

We now wish to discuss forms and cohomology. We have a splitting

$$T_{\mathbf{C}}(M) = T^{1,0}(M) \oplus T^{0,1}(M).$$

By dualizing this we obtain

$$(T_{\mathbf{C}}(M))^* = (T^{1,0}(M))^* \oplus (T^{0,1}(M))^*,$$

and we take wedge products:

$$\bigwedge^k (T^{\mathbf{C}}(M))^* = \bigoplus_{a+b=n, a,b\geq 0} \bigwedge^a (T^{1,0}(M))^* \bigotimes \bigwedge^b (T^{0,1}(M))^*.$$

Let $\mathcal{A}^k(M)$ be the space of smooth complex k-forms on M. Then we can write, locally as,

$$\omega = \sum f_{IJ} dz_I \wedge d\bar{z}_J,$$

where f_{IJ} are smooth functions. Recall that we have a map defined by the exterior differentiation

$$d : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M).$$

Now, $\mathcal{A}^k(M)$ splits, so it is natural to ask what happens to $\mathcal{A}^{a,b}(M)$ under d. Suppose $\omega \in \mathcal{A}^{a,b}(M)$ and we write

$$\omega = \sum_{\#I=a, \#J=b} f_{IJ} dz_I \wedge d\bar{z}_J,$$

then

$$d\omega = \sum_{\#I=a, \#J=b} \left\{ \left(\sum_{i=1}^n \frac{\partial f_{IJ}}{\partial z^i} dz_i \right) \wedge dz_I \wedge d\bar{z}_J + \left(\sum_{i=1}^n \frac{\partial f_{IJ}}{\partial \bar{z}^i} d\bar{z}_i \right) \wedge dz_I \wedge d\bar{z}_J \right\},$$

so in fact, $d\omega \in \mathcal{A}^{a+1,b}(M) \bigoplus \mathcal{A}^{a,b+1}(M)$. Let us write

$$d\omega = \partial\omega + \bar{\partial}\omega$$

where $\partial\omega \in \mathcal{A}^{a+1,b}(M), \bar{\partial}\omega \in \mathcal{A}^{a,b+1}(M)$. Then we can define the Dolbeault cohomology

$$H^{a,b}_{Dol}(M) = ker\bar{\partial}_{a,b}/im\bar{\partial}_{a,b-1}.$$

Let U be an open subset of M, denote by

$$\Omega^p_M(U) := \Gamma(U, \Omega^p_M) := \{\omega \in \mathcal{A}^{p,0}(U), \bar{\partial}\omega = 0\},$$

the set of holomorphic p-forms on U. Ω^p_M is the sheaf of holomorphic p-forms.

Theorem 0.7. *Let X be a compact complex manifold. Then*

$$H^{p,q}_{Dol}(M) = H^q(M, \Omega^p_M),$$

where Ω^p_M is the sheaf of holomorphic p-forms.

Remark: Although we have sheaves $\mathcal{A}^k$, the smooth complex k-forms on M, and $\mathcal{A}^{a,b}$, the smooth (a,b)-forms on M, we have the following statement *Let M be a complex manifold, then $H^p(M,\mathcal{A}^k) = H^p(M,\mathcal{A}^{a,b}) = 0$ for all $p > 0$.* So these sheaves have no interesting cohomology. Here we outline a proof: We only prove that $H^1(M,\mathcal{A}^k) = 0$. Choose a sufficiently general open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of M. Then $\sigma \in H^1(M,\mathcal{A}^k)$, $\sigma = \{\sigma_{\alpha,\beta}\}_{\alpha<\beta}$, we have that $\delta\sigma = 0$ if and only if for all $\alpha < \beta < \gamma$,

$$(\sigma_{\beta,\gamma} - \sigma_{\alpha,\gamma} + \sigma_{\alpha,\beta})|_{U_{\alpha,\beta,\gamma}} = 0.$$

We produce a 0-cycle τ whose image under δ is σ. For this we use the partitions of unity. Pick a partitions of unity $\{\rho_\alpha\}$ subordinate of $\mathcal{U}$, i.e. $\rho_\alpha \in C^\infty(M)$ are such that $\mathrm{supp}(\rho_\alpha) \subset U_\alpha$ for all α and $\sum_\alpha \rho_\alpha = 1$. Define $\tau \in C^0(\mathcal{U},\mathcal{A}^k)$, $\tau = \{\tau_\alpha\}$ by

$$\tau_\alpha = -\sum_\gamma \rho_\gamma \sigma_{\alpha,\gamma},$$

We verify now that $\delta(\tau) = \sigma$. Note that we have extended the $\sigma_{\alpha,\gamma}$ by zero to all of M by multiplying by ρ_γ. Hence we need not worry about domains of definition in the following calculation.

$$(\delta(\tau))_{\alpha,\beta} = \tau_\beta - \tau_\alpha = \sum_\gamma \rho_\gamma(\sigma_{\alpha,\gamma} - \sigma_{\beta,\gamma}) = (\sum_\gamma \rho_\gamma)(\sigma_{\alpha,\beta}) = \sigma_{\alpha,\beta},$$

where the second to last equality holds since σ is a cocycle. Hence $\sigma = 0 \in H^1(M,\mathcal{A}^k)$, and so $H^1(M,\mathcal{A}^k) = 0$.

0.5 Divisors and Line Bundles

Definition 0.8. *An* **invertible sheaf** *is a coherent sheaf $\mathcal{L}$ on M such that each point $x \in M$ has an open neighborhood $U \subset M$ such that $\mathcal{L}(U) \cong \mathcal{O}_U$ as $\mathcal{O}_M$-modules.*

Definition 0.9. *A map $\pi : L \to M$ is said to be a line bundle, if*

(1) $E_p = \pi^{-1}(p)$; $\forall p \in M$, is a linear space with rank 1, i.e. $E_p \cong \mathbf{C}$.

(2) There exists an open covering $\{U_\alpha\}$ of M and bioholomorphic map $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbf{C}$ such that, for all $p \in U_\alpha$, $\phi_\alpha|_{\pi^{-1}(p)} : \pi^{-1}(p) \to \{p\} \times \mathbf{C}$ is a $\mathbf{C}$-linear isomorphism between complex vector space E_p and $\mathbf{C}$.

(U_α, ϕ_α) can be considered as a neighborhood of local coordinations. We call such U_α **the trivialization neighborhood** of L. On $U_\alpha \cap U_\beta \neq \phi$, for $\forall p \in U_\alpha \cap U_\beta$, let $\phi_{\alpha\beta}(p) := \phi_\alpha \circ \phi_\beta^{-1}|\phi_\beta(E_p) : \mathbf{C} \xrightarrow{\phi_\beta^{-1}} E_p \xrightarrow{\phi_\alpha} \mathbf{C}$, is a $\mathbf{C}$-linear isomorphism, i.e. $\phi_{\alpha\beta}(p) \in GL(1,\mathbf{C}) \cong \mathbf{C}^*$, and $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(1,C) \cong \mathbf{C}^*$ is holomorphic. $\phi_{\alpha\beta}$ is called **transitive function** of E.

(3) The $\phi_{\alpha\beta}$ satisfies the compatible conditions provide by the definition of $\phi_{\alpha\beta}$

$$\phi_{\alpha\beta}(p)\phi_{\beta\gamma}(p) = \phi_{\alpha\gamma}(p); \quad p \in U_\alpha \cap U_\beta \cap U_\gamma \tag{1.2}$$

$$\phi_{\alpha\beta}(p) = \phi_{\beta\alpha}(p)^{-1}; \quad p \in U_\alpha \cap U_\beta$$

It turns out that over $\mathbf{P}^n$ there exists essentially only one holomorphic line bundle, *tautological line bundele*, $\mathcal{O}(1)$. Its dual (which some books called it *the univeral line bundle* can be described as follows: *The set* $\mathcal{O}(-1) \subset \mathbf{P}^n \times \mathbf{C}^{n+1}$ *that consists of all pairs* $(l, z) \in \mathbf{P}^n \times \mathbf{C}^{n+1}$ *with* $z \in l$ *forms a natural way a holomorphic line bundle over* $\mathbf{P}^n$. In fact, let $\mathbf{P}^n = \cup_{i=0}^n U_i$ be the standard open covering. A canonical trivialization of $\mathcal{O}(-1)$ over U_i is given by $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbf{C}$, $(l, z) \mapsto (l, z_i)$. The transition maps $\psi_{ij}(l) : \mathbf{C} \to \mathbf{C}$ are given by $w \mapsto \frac{z_i}{z_j} w$, where $l = [z_0 : \cdots : z_n]$, i.e. $\psi_{ij} = \frac{z_i}{z_j}$. Note that the fiber $\mathcal{O}(-1) \to \mathbf{P}^n$ over $l \in \mathbf{P}^n$ is naturally isomorphic to l.

Given an open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of M with U_α being the trivialization neighborhood of L. Then its transition function $\phi_{\alpha\beta} \in \mathcal{O}_M^*(U_\alpha \cap U_\beta)$, where $\mathcal{O}_M^*(U_\alpha$ is the sheaf of nonwhere vanishing holomorphic functions on M. So $\{\phi_{\alpha\beta}\} \in C^1(\mathcal{U}, \mathcal{O}_M^*)$. Further, the compatible conditions (1.2) imply that $\{\phi_{\alpha\beta}\} \in Z^1(\mathcal{U}, \mathcal{O}_M^*)$. We we get a map $L \mapsto [\{\phi_{\alpha\gamma}\}] \in H^1(\mathcal{U}, \mathcal{O}_M^*)$. In this way, we can prove the following important statement: *There is one-to-one correspondence between the equivalent classes of holomorphic line bundles on* M *and the elements of the cohomology group* $H^1(M, \mathcal{O}_M^*)$.

A *holomorphic section* s of L on an open set $U \subset M$ is a holomorphic map $s : U \to L$ such that $p \circ s = id_U$.

Denote

$$\Gamma(U, L) = \{s; s : U \to L \text{ holomorphic section}\}.$$

This defines a coherent analytic sheaf (of sections) $\mathcal{L}$ over X by $\mathcal{L}(U) = \Gamma(U, L)$. It is **an invertible sheaf** since

$$\mathcal{L}(U_\alpha) \cong \mathcal{O}_{U_\alpha}.$$

Conversely, let $\mathcal{L}$ be an invertible sheaf, and let $\phi_\alpha : \mathcal{L}(U_\alpha) \cong \mathcal{O}_{U_\alpha}$ be the local trivialiazrions. Then $g_{\alpha,\beta} = \phi_\alpha \circ \phi_\beta^{-1}$ gives the line bundle L. Hence, we also call **invertible sheaf** as **line bundle**.

The concept of line bundle is intimitely related to the concept of **divisors**, which originated from the Riemann surfaces. On a Riemann surface, poles and zeros of meromorphic functions are isolated points. We use $p_1, \cdots p_n$ to denote these isolate points. Then the formal sum, $\sum n(p_i) p_i$, is called a divisor, where $n(p_i) \in Z$. Those $n(p_i) \in Z^+$ denote the multiplicities of the zeros p_i, and those $n(p_i) \in Z^-$ denote the multiplicities of the poles p_i. So, in fact, $\sum_i n(p_i) p_i$ reflects a meromorphic function with the given poles and zeros, counting multiplicities.

For a complex manifold M, the divisor is a complex submanifold with codimention 1, which is locally defined by the set of zeros of a holomorphic function. Alternatively (Weil's divisor)

Definition 0.10. *A divisor* D *on* M *is a formal linear combination*

$$D = \sum a_i [Y_i]$$

where $Y_i \subset M$ *irreducible hypersurfaces and* a_i *are integers. The divisor group* $Div(X)$ *is the set of all divisors endowed with the natural group structure. A divisor* D *is called effective if* $a_i \geq 0$ *for all* i*.*

Let D be a divisor on M, and $\{U_i\}_{i\in I}$ be an open covering of M such that on each U_i; $i \in I$, $D \cap U_i = \{f_i = 0\}$, where f_i is a holomorphic function on U_i. When $U_i \cap U_j \neq \phi$

$$\phi_{ij} := \frac{f_i}{f_j} \quad U_i \cap U_j,$$

then $\phi_{ij} \neq 0$ on $U_i \cap U_j$ and $\phi_{ij} \cdot \phi_{ji} = 1$; on $U_i \cap U_j$, $\phi_{ij}\phi_{jk}\phi_{ki} = 1$ on $U_i \cap U_j \cap U_k$, so $\{\phi_{ij}\}_{i\in I}$ is a transitive function, which defines a line bundle L. We call L the line bundle associated to the divisor D, and denote it by $L = [D]$. If D is defined by $D \cap U_i = \{f_i = 0\}$, where $\{U_i\}_{i\in I}$ is an open covering of M and f_i is holomorphic function, then $\{f_i\}_{i\in I}$ is a holomorphic section over M, i.e. $f \in \Gamma(M, [D])$

$$f \mid U_i = f_i.$$

Obviously the zeros of f is just the divisor D.

We need to point that the $[D]$ is unique in the isomorphic sense of line bundles. If there is another system of holomorphic functions defining D, then $\frac{f_i}{f_i'} \neq 0$ on U_i; $\forall i \in I$, then

$$u_i = \frac{f_i}{f_i'} : u_i \longrightarrow \mathbf{C}^* = \mathbf{C}\backslash\{\mathbf{0}\}$$

so that

$$\phi_{ij} = \frac{f_i}{f_j} = \frac{u_i}{u_j} \cdot \frac{f_i'}{f_j'} = \frac{u_i}{u_j} = \phi_{ij}'.$$

By theorem 10.15 the line bundles defined by $\{\phi_{ij}\}$ and $\{\phi_{ij}'\}$ are equivalent.

Let's take $H : a_0 z_0 + \cdots + a_n z_n = 0$ be a hyperplane in $\mathbf{P}^n$. let $\mathbf{P}^n = \cup_{i=0}^n U_i$ be the standard open covering. Then on U_i, we have $f_i = a_0 \frac{z_0}{z_i} + \cdots + a_n \frac{z_n}{z_i}$, hence $\phi_{ij} := \frac{f_i}{f_j} = \frac{z_j}{z_i} \quad U_i \cap U_j$. So the line bundle associated to H is $\mathcal{O}(1)$.