Riemannian Geometry, Homework #1 (due date: Tuesday, Sept. 8)

1. Let
\[ \sigma(u, v) = (\sin u \cos v, \sin u \sin v, \cos u), \quad 0 < u < \pi, 0 < v < 2\pi \]
be a parametrization of the unit sphere
\[ S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}. \]
Fix an angle \( 0 < \theta_0 < \pi \) and consider the parallel (in short, we just write \( u = \theta_0 \)) on the unit sphere
\[ \alpha(t) = (\sin \theta_0 \cos t, \sin \theta_0 \sin t, \cos \theta_0), \quad 0 < t < 2\pi. \]

(i) Sketch the curve \( \alpha \).

(ii) Calculate the normal curvature of \( \alpha \).

2. Show that the normal curvature of any curve on a sphere of radius \( r \) is \( \pm 1/r \).

3. Show that if a curve on a surface has zero normal and geodesic curvature everywhere, then it is part of a straight line.

4. Find the Gauss curvature, mean curvature, principal curvatures and the corresponding principal directions of the following surfaces

   (a) \( \sigma(u, v) = (a(u + v), b(u - v), 4uv) \) where \( a \) and \( b \) are constant.

   (b) The cylinder: \( \sigma(u, v) = (a \cos u, a \sin u, v) \).

5. Calculate the Christoffel symbols for the surface \( z = f(x, y) \).

6. Assume that the surface \( \sigma(u, v) \) has its first fundamental form as
\[ E = \frac{4}{(1 + u^2 + v^2)^2}, \quad F = 0, \quad G = \frac{4}{(1 + u^2 + v^2)^2}. \]
Prove its Gauss curvature \( K \equiv 1 \).