Riemannian Geometry, Key to Homework #1

1. Let \( \sigma(u, v) = (\sin u \cos v, \sin u \sin v, \cos u), \quad 0 < u < \pi, 0 < v < 2\pi \)
be a parametrization of the unit sphere
\[ S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1 \}. \]
Fix an angle \( 0 < \theta_0 < \pi \) and consider the parallel (in short, we just write \( u = \theta_0 \)) on
the unit sphere
\[ \alpha(t) = (\sin \theta_0 \cos t, \sin \theta_0 \sin t, \cos \theta_0), \quad 0 < t < 2\pi. \]
(i) Sketch the curve \( \alpha \).
(ii) Calculate the normal curvature of \( \alpha \).

Solution: (ii) Method 1: By the calculation,
\[ n = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \pm (\sin u \cos v, \sin u \sin v, \cos u). \]
On the other hand, \( \alpha'(t) = (-\sin \theta_0 \sin t, \sin \theta_0 \cos t, 0) \). It is not unit-speed, its arc-length parameterization is
\[ \alpha(s) = (\sin \theta_0 \cos (s/ \sin \theta_0), \sin \theta_0 \sin (s/ \sin \theta_0), 0). \]
Hence,
\[ \alpha''(s) = \frac{1}{\sin \theta_0} (-\cos (s/ \sin \theta_0), -\sin (s/ \sin \theta_0), 0). \]
Write \( \alpha(s) = \sigma(u(s), v(s)) \) we find
\[ u(s) = \theta_0, \quad v(s) = \frac{s}{\sin \theta_0}. \]
Hence, the restriction of \( n \) to \( \alpha(s) \) is
\[ n(s) = \pm (\sin \theta_0 \cos (s/ \sin \theta_0), \sin \theta_0 \sin (s/ \sin \theta_0), \cos \theta_0). \]
By the formula, the normal curvature of \( \alpha \) is
\[ \kappa_n(s) = n(s) \cdot \alpha''(s) = \pm \cos^2 (s/ \sin \theta_0) + \sin^2 (s/ \sin \theta_0) = \pm 1. \]
Method 2: We can also use Theorem 4.1.1., i.e. (4.1.5) to calculate the normal curvature, where \( \mathbf{v} = \mathbf{\alpha}'(t) \), where \( \mathbf{\alpha} \) must be a unit-speed curve. To do so, we need to calculate the second fundamental form \( e, f, g \). By a simple calculation, we have

\[
e = -1, \quad f = 0, \quad g = -\sin^2 u.
\]

On the other hand, as we did above, after re-parameterized by arc-length parameter, we get

\[
\mathbf{\alpha}(s) = (\sin \theta_0 \cos(s/\sin \theta_0), \sin \theta_0 \sin(s/\sin \theta_0), 0).
\]

Write \( \mathbf{\alpha}(s) = \mathbf{\sigma}(u(s), v(s)) \) we find

\[
u(s) = \theta_0, \quad v(s) = \frac{s}{\sin \theta_0}.
\]

Thus, \( u'(s) = 0, v'(s) = 1/\sin \theta_0 \). Hence

\[
\kappa_n(s) = II(\mathbf{\alpha}'(s), \mathbf{\alpha}''(s)) = e(u'(s))^2 + 2fu'(s)v'(s) + g(v'(s))^2 = -\sin^2 \theta_0 \left( \frac{1}{\sin \theta_0} \right)^2 = -1.
\]

Since the principal normal vector can have two directions, we have \( \kappa_n(s) = \pm 1 \).

2. Show that the normal curvature of any curve on a sphere of radius \( r \) is \( \pm 1/r \).

Solution: Let \( M = \{(x, y, z) \mid x^2 + y^2 + z^2 = r^2 \} \). Then its unit normal is just its normalized position vector, i.e.

\[
n = \pm \frac{1}{\sqrt{x^2 + y^2 + z^2}}(x, y, z) = \pm \frac{1}{r}(x, y, z).
\]

Let \( \mathbf{\alpha}(t) \) be a curve on the sphere \( M \), then the restriction of \( \mathbf{n} \) to \( \mathbf{\alpha}(t) \) is

\[
n(t) = \pm \frac{1}{r}\mathbf{\alpha}(t).
\]

Therefore, by the formula, the normal curvature of \( \mathbf{\alpha} \) is

\[
\kappa_n = n(t) \cdot \mathbf{\alpha}''(t) = \pm \frac{1}{r}\mathbf{\alpha}(t) \cdot \mathbf{\alpha}''(t).
\]

Since \( \mathbf{\alpha} \) is on the sphere, \( \mathbf{\alpha} \cdot \mathbf{\alpha} = r^2 \). By differentiate it on both sides, we have

\[
2\mathbf{\alpha} \cdot \mathbf{\alpha}' = 0.
\]

Differentiating it on both sides again yields \( \mathbf{\alpha}' \cdot \mathbf{\alpha}' + \mathbf{\alpha} \cdot \mathbf{\alpha}'' = 0 \). Since \( \mathbf{\alpha} \) is unit-speed, \( \mathbf{\alpha}' \cdot \mathbf{\alpha}' = 1 \). Hence \( \mathbf{\alpha} \cdot \mathbf{\alpha}'' = -1 \). Therefore,

\[
\kappa_n = \pm \frac{1}{r}\mathbf{\alpha}(t) \cdot \mathbf{\alpha}''(t) = \pm \frac{1}{r}.
\]
3. Show that if a curve on a surface has zero normal and geodesic curvature everywhere, then it is part of a straight line.

**Solution:** Since \( \kappa^2 = \kappa_n^2 + \kappa_g^2 \), The condition that a curve on a surface has zero normal and geodesic curvature everywhere implies that its curvature is identically zero. So, by the theorem, it is part of a straight line.

4. Find the Gauss curvature, mean curvature, principal curvatures and the corresponding principal directions of the following surfaces

(a) \( \sigma(u, v) = (a(u + v), b(u - v), 4uv) \) where \( a \) and \( b \) are constant.

(b) The cylinder: \( \sigma(u, v) = (a \cos u, a \sin u, v) \).

**Solution:** (a) \( \sigma_u = (a, b, 4v), \sigma_v = (a, -b, 4u), \sigma_{uu} = (0, 0, 0), \sigma_{vv} = (0, 0, 0), \sigma_{uv} = (0, 0, 4) \).

The unit normal is

\[
\mathbf{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = \frac{(2b(u + v), 2a(v - u), -ab)}{(4b^2(u + v)^2 + 4a^2(u - v)^2 + a^2b^2)^{1/2}}.
\]

The first fundamental form is

\[
E = \sigma_u \cdot \sigma_u = a^2 + b^2 + 16v^2, \quad F = \sigma_u \cdot \sigma_v = a^2 - b^2 + 16uv, \quad G = \sigma_v \cdot \sigma_v = a^2 + b^2 + 16u^2.
\]

The second fundamental form is

\[
e = \mathbf{n} \cdot \sigma_{uu} = 0, \quad f = \mathbf{n} \cdot \sigma_{uv} = \frac{-4ab}{(4b^2(u + v)^2 + 4a^2(u - v)^2 + a^2b^2)^{1/2}}, \quad g = \mathbf{n} \cdot \sigma_{vv} = 0.
\]

\[ EG - F^2 = 16a^2(u - v)^2 + 16b^2(u + v)^2 + 4a^2b^2. \]

The Gauss curvature is

\[
K = \frac{eg - f^2}{EG - F^2} = \frac{-4a^2b^2}{(4b^2(u + v)^2 + 4a^2(u - v)^2 + a^2b^2)^2}.
\]

The mean curvature is

\[
H = \frac{eG - 2fF + gE}{2(EG - F^2)} = \frac{ab(a^2 - b^2 + 16uv)}{(4b^2(u + v)^2 + 4a^2(u - v)^2 + a^2b^2)^{3/2}}.
\]

To calculate the principal curvatures, note that

\[
\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix},
\]

\[ 3 \]
So the matrix of the shape operator $S_P$ with respect to the basis $\{\sigma_u, \sigma_v\}$ is
\[
A = \mathcal{F}_1^{-1} \mathcal{F}_{II} = \frac{1}{(EG - F^2)^{3/2}} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} 0 & -8ab \\ -8ab & 0 \end{pmatrix} = \frac{8ab}{(EG - F^2)^{3/2}} \begin{pmatrix} F & -G \\ -E & F \end{pmatrix}.
\]

Write $\mu = \frac{8ab}{(EG - F^2)^{3/2}}$, then
\[
\det(A - \lambda I) = (\mu F - \lambda)^2 - \mu^2 EG.
\]

Setting $\det(A - \lambda I) = 0$, i.e. $(\mu F - \lambda)^2 - \mu^2 EG = 0$, we get
\[
\mu F - \lambda = \pm \mu \sqrt{EG}.
\]

Hence, the eigenvalues (principal curvatures) are
\[
\kappa_1 = \mu (F + \sqrt{EG}), \kappa_2 = \mu (F - \sqrt{EG}),
\]
where $\mu, E, F, G$ are given as above. To get the principal directions, for $\kappa_1 = \mu (F + \sqrt{EG})$, we solve $(A - \kappa_1 I)v = 0$, i.e.,
\[
\begin{pmatrix} -\mu \sqrt{EG} & -\mu G \\ -\mu E & -\mu \sqrt{EG} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

We get (we need normalize it to get a unit-vector!!!)
\[
\begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} = \frac{1}{\sqrt{1 + \frac{E}{G}}} \begin{pmatrix} 1 \\ -\sqrt{\frac{E}{G}} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{G}{E+G}} \\ -\sqrt{\frac{E}{E+G}} \end{pmatrix}.
\]

Similarly,
\[
\begin{pmatrix} \xi_2 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{G}{E+G}} \\ \sqrt{\frac{E}{E+G}} \end{pmatrix}.
\]

Hence the principal directions are
\[
e_1 = \sqrt{\frac{G}{E+G}} \sigma_u - \sqrt{\frac{E}{E+G}} \sigma_v, \quad e_2 = \sqrt{\frac{G}{E+G}} \sigma_u + \sqrt{\frac{E}{E+G}} \sigma_v,
\]
where $\mu = \frac{8ab}{(EG - F^2)^{3/2}}$, and $E, F, G$ are given as above.

(ii) By direct calculation, the first fundamental form is
\[
E = a^2, F = 0, G = 1,
\]
and the second fundamental form is

\[ e = -a, f = 0, g = 0. \]

Hence the matrix of the shape operator \( S \) with respect to the basis \( \{\sigma_u, \sigma_v\} \) is

\[ A = \mathcal{F}_I^{-1} \mathcal{F}_{II} = \begin{pmatrix} -1/a & 0 \\ 0 & 0 \end{pmatrix}. \]

Solving the equation \( \det(A - \lambda I) = 0 \), we get the principal curvatures \( \kappa_1 = 0, \kappa_2 = -1/a \). To get the principal directions, for \( \kappa_1 = 0 \), we solve \((A - \kappa_1 I)v = 0\), i.e.,

\[ \begin{pmatrix} -1/a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

We get (we need normalize it to get a unit-vector!!!) \( \xi_1 = 0, \eta_1 = 1 \), so \( e_1 = \sigma_v \). Similarly, for \( \kappa_2 = -1/a \), we get \( \xi_1 = 1, \eta_1 = 0 \), so \( e_2 = \sigma_u \).

The Gauss curvature is \( K = \kappa_1 \kappa_2 = 0 \) and the mean curvature is

\[ H = \frac{\kappa_1 + \kappa_2}{2} = -\frac{1}{2a}. \]

5. Calculate the Christoffel symbols for the surface \( z = f(x, y) \).

**Solution:** Let \( \sigma(u, v) = (u, v, f(u, v)) \). Then \( \sigma_u = (1, 0, f_u), \sigma_v = (0, 1, f_v) \). Hence

\[ E = 1 + f_u^2, \quad F = f_u f_v, \quad G = 1 + f_v^2. \]

\[ E_u = 2f_u f_{uu}, \quad E_v = 2f_u f_{uv}, \quad F_u = f_{uu} f_u + f_{uv} f_v, \quad F_v = f_{uv} f_u + f_{vv} f_v, \quad G_u = 2f_v f_{uv}, \quad G_v = 2f_v f_{vv}. \]

Hence,

\[ \Gamma^1_{11} = \frac{f_u f_{uu}}{1 + f_u^2 + f_v^2}, \]

\[ \Gamma^1_{12} = \Gamma^1_{21} = \frac{f_u f_{uv}}{1 + f_u^2 + f_v^2}, \]

\[ \Gamma^1_{22} = \frac{f_v f_{vv}}{1 + f_u^2 + f_v^2}, \]

\[ \Gamma^2_{11} = \frac{f_v f_{uu}}{1 + f_u^2 + f_v^2}, \]

\[ \Gamma^2_{12} = \Gamma^2_{21} = \frac{f_v f_{uv}}{1 + f_u^2 + f_v^2}. \]
\[ \Gamma_{22}^2 = \frac{f_v f_{vv}}{1 + f_u^2 + f_v^2}. \]

6. Assume that the surface \( \sigma(u, v) \) has its first fundamental form as

\[
E = \frac{4}{(1 + u^2 + v^2)^2}, \quad F = 0, \quad G = \frac{4}{(1 + u^2 + v^2)^2}.
\]

Prove its Gauss curvature \( K \equiv 1 \).

**Solution:** Use the following formula directly:

\[
K = \frac{-1}{2\sqrt{EG}} \left( \left( \frac{E_v}{\sqrt{EG}} \right)_v + \left( \frac{G_u}{\sqrt{EG}} \right)_u \right).
\]