Abstract Surfaces (2-dimensional Riemannian manifold)

Recall (Ch. 1 of Oprea)

Def: A coordinate patch (or parametrization)

is a one-to-one regular mapping

\[ \mathbf{z}: D \rightarrow \mathbb{R}^3 \]

of an open set \( D \subseteq \mathbb{R}^2 \).

Def: A surface in \( \mathbb{R}^3 \) is a subset \( M \subseteq \mathbb{R}^3 \)

such that each point of \( M \) has a neighborhood (in \( M \)) contained in the image of some coordinate patch

\[ \mathbf{z}: D \rightarrow M \subseteq \mathbb{R}^3 \quad D \subseteq \mathbb{R}^2 \].
To define 2-dim'l abstract surfaces, we need to remove the "$\mathbb{R}^3$" in the definition. The following are important ingredients that we want to keep:

- $M = \text{set with concept of continuity}$
- $\forall p \in M, \exists \text{"subset" } U \text{ of } M$ such that $p \in U$ and we can assign coordinates on $U$
• Change of coordinates are differentiable.

What do we mean by coordinates (in 2-dimensions)?

\[ \forall p \rightarrow \text{a unique pair } (u, v) \]

of numbers

Hence assigning coordinates means

\[ \exists \text{ a "homeomorphism" on } U \]

(i.e. continuous map with continuous inverse)

to an open subset \( D \) in \( \mathbb{R}^2 \).
Therefore, we are forced to make the following definition: (Not really the standard one)

**Def**: \( M = \text{Abstract (differentiable) surface} \)

If \( M = \bigcup_{\alpha \in \Lambda} B_{\alpha}(D_{\alpha}) \) where

- \( D_{\alpha} \) open in \( \mathbb{R}^2 \)
- \( B_{\alpha} : D_{\alpha} \to M \) is 1-1 (cts & inverse from \( B_{\alpha}(D_{\alpha}) \)) is also cts.

So that if \( B_{\alpha}(D_{\alpha}) \cap B_{\beta}(D_{\beta}) \neq \emptyset \),
Then
\[ x^{-1}_\beta \circ x_\alpha : x^{-1}_\alpha (x_\alpha(D_\alpha) \cap x_\beta(D_\beta)) \to x^{-1}_\beta (x_\alpha(D_\alpha) \cap x_\beta(D_\beta)) \]
is differentiable.
Remark: This definition formalizes the idea that an "abstract surface" is a union of coordinate patches (so that the change of coordinates are differentiable).

Remark: The formal definition needs the concept of Hausdorff space.
The above definition doesn't include the geometric concepts such as "length", "distance", "area" etc.

These concepts need the introduction of Riemannian metric.

Let's check the situation for surfaces in $\mathbb{R}^3$:

- length of a curve $L = \int_a^b |x'(t)| \, dt$
where \( \alpha'(t) \in T_{\alpha(t)}M \)

is the tangent vector of the curve \( \alpha \)

which belongs to the tangent plane \( T_{\alpha(t)}M \)
to \( M \) at the point \( \alpha(t) \).

\[ |\alpha'(t)| = \text{length of the tangent vector} \]

• Distance of 2 points \( p, q \in M \)

\[ = L(\alpha), \text{ where } \alpha \text{ is a curve in } M \]

connecting \( p \) & \( q \) with shortest length.
Given $|\mathbf{a}'(t)|$, a possible $\mathbf{a}'(t) \in T_{\mathbf{a}(t)} \mathcal{M}$, one can define lengths & distance on $\mathcal{M}$.

- For surfaces in $\mathbb{R}^3$, $|\mathbf{a}'(t)| = \text{length of the } 2\text{-vector } \mathbf{a}'(t)$. But what is $\mathbf{a}'(t)$ for a general abstract surface $\mathcal{M}$?
One simple define the collection of all \( \mathcal{L}(x) \) to be the tangent plane \( T_{\mathcal{L}(x)} M \) to \( M \) at \( \mathcal{L}(x) \).

\( \text{Caution: } \exists \text{ many curves } \beta(x) \text{ s.t. } \beta'(t) = \mathcal{L}'(t). \)

Formal definition of \( T_p M, p = \mathcal{L}(x) \), needs to take care of this.
Remark: Of course, we need to define (diff.) curve $\gamma$ on $M$ first.

But this is easy: just mapping from an interval $[a, b]$ to $M$ such that within each coordinate patch $\mathcal{A}(D)$,

$$\gamma(t) = (u(t), v(t))$$

($u, v$ differentiable in $t$)
To define $\mathbf{d}'(x)$ or generally $\mathbf{d}'$ for $\mathbf{d} \in T_pM$, we recall the natural basis $\{\mathbf{e}_u, \mathbf{e}_v\}$ for the coordinate patch $\mathcal{X}(u, v)$:

$$\mathbf{e}_u = \frac{\partial \mathbf{X}}{\partial u}, \quad \mathbf{e}_v = \frac{\partial \mathbf{X}}{\partial v}$$

and for $\mathbf{X}(x) = (u(x), v(x))$

$$\mathbf{X}'(x) = u' \frac{\partial \mathbf{X}}{\partial u} + v' \frac{\partial \mathbf{X}}{\partial v}$$
\[ |\lambda'(x)|^2 = \left( \frac{\partial \lambda}{\partial u} \right)^2 u'^2 + 2 \left( \frac{\partial \lambda}{\partial u}, \frac{\partial \lambda}{\partial v} \right) u' v' + \left( \frac{\partial \lambda}{\partial v} \right)^2 v'^2 \]

In index form:

\[ |\lambda'(x)|^2 = g_{ij} \left( \frac{du^1}{dt} \right)^2 + 2 g_{12} \frac{du^1}{dt} \frac{du^2}{dt} + g_{22} \left( \frac{du^2}{dt} \right)^2 \]

\[ = \sum_{i,j=1}^{2} g_{ij} (u^1_{ix}) (u^2_{ix}) \frac{du^1}{dx} \frac{du^2}{dx} \]

\[ (u^1(x), u^2(x)) = \mathbf{u} = \lambda(x) \]
And for a general \( V = (V^1, V^2) \in T_pM \),

\[
|V|^2 = \sum_{i,j=1}^{2} g_{\bar{i}\bar{j}}(p) V^{\bar{i}} V^{\bar{j}}
\]

Therefore, length of tangent vectors in \( T_pM \) is determined by

\[
\left( g_{\bar{i}\bar{j}}(p) \right)
\]
Note that \((g_{ij}(p))\) satisfies

1. \(g_{ij}(p) = g_{ij}(u,v) \in C^\infty\) differentiable in \((u,v)\)

2. \((g_{ij})\) is symmetric \((g_{ij} = g_{ji})\)

3. \((g_{ij}) > 0\) (positive definite)

\[
(\text{i.e. } (V_1, V_2) (g_{ij}) (V_1^\top V_2) > 0, \quad V_1, V_2)
\]
Reversing the process, we see that if we are given, for each coordinate patch, the metric \((g_{ij}(p))\) satisfying (1), (2), (3), we can define (within a patch)

\[
\left\{
\begin{array}{l}
\text{length of tangent vector} \\
\text{length of curves} \\
\text{distance between 2 points!}
\end{array}
\right.
\]
Furthermore, recall that

\[ |\mathbb{R}^n \times \mathbb{R}^m| \, du dv = \sqrt{\det(g_{ij}(u,v))} \, du dv \]

is the area element of a surface in \( \mathbb{R}^3 \), we can define

* area of a region on an abstract surface provided \( (g_{ij}) \) is given.
The above is the great observation by Riemann & $(g_{ij})$ is called the Riemannian metric nowadays for an abstract surface.

Remark: There is one more important issue to handle: How $(g_{ij})$ changes in different coordinates?
In order to have a well-defined length $|V|$ for $V \in T_pM$, our definition of $(G_{ij}(p))$ should be compatible with respect to different coordinate patches $\mathcal{X}(uv)$ and $\mathcal{X}(\tilde{u}, \tilde{v})$ around $p$.

Let $\alpha(t)$ be a curve passing through $p$. 
In $\mathbb{R}(u,v)$ coordinates,

$$\mathbf{d}(t) = (u(t), v(t))$$

(not more precisely $\mathbf{d}(t) = \mathbb{R}(u(t), v(t))$)

In $\mathbb{R}(\widehat{u}, \widehat{v})$ coordinates,

$$\mathbf{d}(t) = (\widehat{u}(t), \widehat{v}(t))$$

$$\mathbf{d}(t) = \mathbb{R}(\widehat{u}(t), \widehat{v}(t))$$

$$\Rightarrow$$

$$\mathbf{d}'(t) = \frac{\partial \mathbf{d}}{\partial u} \frac{du}{dt} + \frac{\partial \mathbf{d}}{\partial v} \frac{dv}{dt}$$
\[
\tilde{x} = \frac{\partial \tilde{x}}{\partial \tilde{u}} \frac{d \tilde{u}}{dt} + \frac{\partial \tilde{x}}{\partial \tilde{v}} \frac{d \tilde{v}}{dt}
\]
Now \((\widetilde{u}, \widetilde{v}) = (\overline{X}^{-1} \circ \overline{X})(u, v)\)

is the change of coordinate \&

Let denote it by

\[ F = \overline{X}^{-1} \circ \overline{X}. \]

Then

\[ \begin{aligned}
    \widetilde{u} &= F^1(u, v) \\
    \widetilde{v} &= F^2(u, v)
\end{aligned} \]

From

\[ \overline{X}(u, v) = \overline{X} \circ (\overline{X}^{-1} \circ \overline{X})(u, v) \]
we have $\bar{X} = \tilde{X} \circ F$

$$\Rightarrow \frac{\partial \tilde{X}}{\partial u} = \frac{\partial \tilde{X}}{\partial \tilde{u}} (F) \frac{\partial F}{\partial u} + \frac{\partial \tilde{X}}{\partial \tilde{u}} (F) \frac{\partial \tilde{F}}{\partial u}$$

In index form

$$\bar{X}_i = \sum_{j=1}^{\tilde{S}} \tilde{X}_j \frac{\partial F_j}{\partial u_i}$$
Hence
\[ g_{ij} = \langle \tilde{\mathbf{S}}_i, \tilde{\mathbf{S}}_j \rangle \]
\[ = \sum_{k=1}^{2} \sum_{l=1}^{2} \left( \frac{\partial F^k}{\partial u^i} \right) \left( \frac{\partial F^l}{\partial u^j} \right) \]

\[ \Rightarrow g_{ij} = \sum_{k, l=1}^{2} \tilde{g}_{kl} \frac{\partial F^k}{\partial u^i} \frac{\partial F^l}{\partial u^j} \]

Note that \[ p^k = \tilde{u}^k \], and usually write
\[ g_{ij} = \sum_{k, l=1}^{2} \tilde{g}_{kl} \frac{\partial \tilde{u}^k}{\partial u^i} \frac{\partial \tilde{u}^l}{\partial u^j} \]
In matrix form

\[
(g_{i,j}) = \left( \frac{\partial \tilde{u}}{\partial (u,v)} \right)^T \left( \frac{\partial \tilde{v}}{\partial (u,v)} \right) \left( \frac{\partial \tilde{u}}{\partial (u,v)} \right)
\]

where

\[
\frac{\partial \tilde{u}}{\partial (u,v)} = \left( \begin{array}{cc}
\frac{\partial \tilde{u}}{\partial u} & \frac{\partial \tilde{u}}{\partial v} \\
\frac{\partial \tilde{v}}{\partial u} & \frac{\partial \tilde{v}}{\partial v}
\end{array} \right)
\]

the Jacobian matrix of the change of coordinates: \((\tilde{u}, \tilde{v}) = F(u, v)\).
In notation of differentials:

note that

\[ d\tilde{u}^k = \sum_{i=1}^2 \frac{\partial \tilde{u}_k}{\partial u^i} \, du^i, \]

we see that the expression

\[ \sum_{k, l=1}^2 \tilde{g}_{k e} \, d\tilde{u}^k \otimes d\tilde{u}^l \]

\[ = \sum_{k, l=1}^2 \tilde{g}_{k e} \left( \sum_{i} \frac{\partial \tilde{u}_h}{\partial u^i} \, du^i \right) \otimes \left( \sum_{j} \frac{\partial \tilde{u}_l}{\partial u^j} \, du^j \right) \]

\[ = \sum_{i, j} \left( \sum_{k, l=1}^2 \tilde{g}_{k e} \frac{\partial \tilde{u}_h}{\partial u^i} \frac{\partial \tilde{u}_l}{\partial u^j} \right) \, du^i \otimes du^j \]
\[= \sum \hat{g}_{ij} \, du^i \otimes du^j \] by the change of coordinates formula for \( g_{ij} \)

We see that the expression

\[
g = \sum_{i,j=1}^{2} \hat{g}_{ij} \, du^i \otimes du^j
\]

is invariant under the change of coordinates.

And we call "\( g \)" a Riemannian metric (tensor).
Therefore, in order to study "geometry" on an abstract surface $M$, we need to assign a Riemannian metric (tensor) $g$ on $M$, i.e. $(g_{ij})$ for each coordinates satisfying
\[ g_{ij} = \sum_{k,l=1}^{2} \frac{\partial u^k}{\partial u^i} \frac{\partial u^l}{\partial u^j} \]
under change of coordinates $(\tilde{u}, \tilde{v}) \leftrightarrow (u, v)$.
Def. A Riemannian manifold of 2-dimension $(M, g)$ is an abstract surface together with a Riemannian metric (tensor).

$\text{deg}: (\mathbb{R}^2, \text{Euclidean})$

We need only 1 single coordinate patch, namely $\mathbb{R}^2$ itself. On this
patch \( \mathcal{X}(u,v) = (u,v) \in \mathbb{R}^2 \),
we define
\[
(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Then \( g = du \otimes du + dv \otimes dv \)
\( u \quad g = du^2 + dv^2 \) (in short)
is a Riemannian metric. (no other coordinate patch.)
With respect to this $g = du^2 + dv^2$, the length of a tangent vector

$$ \mathbf{\alpha}'(t) = (U'(t), V'(t)) $$

of a curve $\mathbf{\alpha}(t) = (U(t), V(t)) \in \mathbb{R}^2$ is

$$ |\mathbf{\alpha}'|^2 = g_{11} U'^2 + 2g_{12} U' V' + g_{22} V'^2 $$

$$ = U'^2 + V'^2 \quad (g_{11} = g_{22} = 1) \quad g_{12} = 0 $$

The Euclidean length!
eq: \( \mathbb{R}^2_+ = \{(u,v) \in \mathbb{R}^2 : u > 0 \} \)

\( \mathbb{R}^2_+ \) itself is a patch. We define

\[
g = \frac{du^2 + dv^2}{u^2} \quad (u > 0)
\]

Then any curve \( \gamma(t) = (u(t), v(t)) \),

\[
|\gamma'|^2_g = \frac{u'^2 + v'^2}{u^2} \quad \text{non-Euclidean}
\]
Curvature of a 2-dimensional Riemannian manifold.

By Gauss theorem, the Gauss curvature can be expressed in terms of $G_{ij}$ and their derivatives only. Therefore, one can define Gauss curvature (for 2-dim) by the Gauss formula:
\[ K = \frac{1}{\text{det}(g_{ij})} \sum_{k=1}^{2} g^{kk} \left[ \frac{\partial \phi}{\partial u^k} \frac{\partial \phi}{\partial u^l} \frac{\partial \phi}{\partial u^m} \sum_{s=1}^{2} (n_{s1} n_{s2} - n_{s2} n_{s1}) \right] \]

where

\[ \nabla^k_{ij} = \frac{1}{2} \sum_{l=1}^{2} g^{kl} \left( \frac{\partial \phi_{ij}}{\partial u^l} + \frac{\partial \phi_{il}}{\partial u^j} - \frac{\partial \phi_{lj}}{\partial u^i} \right) \]

are the Christoffel symbols.
In particular, we have the special formula for

\[(g_{i\,j}^-) = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}\]

\[K = -\frac{1}{2\sqrt{g_{11}g_{22}}} \left[ \frac{\partial}{\partial u^2} \left( \frac{\partial g_{11}}{\partial u^2} \right) + \frac{\partial}{\partial u'} \left( \frac{\partial g_{22}}{\partial u'} \right) \right]\]

\[(g_{i\,j}^+) = \begin{pmatrix} e^{2\gamma} & 0 \\ 0 & e^{2\gamma} \end{pmatrix}\]
\[ K = -e^{-2f} \Delta f \]

Where \[ \Delta f = \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} \]

\[ \text{For } \left( \mathbb{R}^2 \right)^+, \quad g = \frac{du^2 + dv^2}{u^2} \]

we have
\[ g = \begin{pmatrix} \frac{1}{u^2} & 0 \\ 0 & \frac{1}{v^2} \end{pmatrix} = \begin{pmatrix} e^{2f} & 0 \\ 0 & e^{2f} \end{pmatrix} \]

with \[ f = -\log u \]

Then \[ \Delta f = -\frac{\partial^2}{\partial u^2} \log u \]
\[ \frac{\partial}{\partial \nu} \left( \frac{1}{\nu} \right) = \frac{1}{\nu^2} \]

\[ \Rightarrow \quad \text{Curvature} \quad K = -e^{-2f} \Delta f \]

\[ = -\left( \nu^2 \right) \cdot \frac{1}{\nu^2} = -1 \]

\[ \therefore \quad (\mathbb{R}_+^2, \ g = \frac{du^2 + dv^2}{\nu^2}) \text{ is the hyperbolic geometry.} \]

To see this is the case, one must
check $L = 0 \Sigma$ is in fact never reached by a point inside $\mathbb{R}^2_+$, i.e. any path from inside $\mathbb{R}^2_+$ to the boundary is of infinite length.

Example:

\[ L(t) = (1-t) \hat{i} \]
\[ L'(t) = (1-t) \hat{j} \]

Then \[ L'(t) = -\hat{i} \]

(i.e. \( \langle u', u' \rangle = (0, -1) \))
\[ \lambda'(x)^2 = \frac{u^2 + v^2}{u^2} = \frac{(-1)^2}{(1-x)^2} \]

\[ = \frac{1}{(1-x)^2} \]

\[ \int_0^1 \lambda'(x) \, dx = \int_0^1 \frac{1}{1-x} \, dx \]

\[ = \lim_{x \to 1} \log \frac{1}{1-x} = \infty. \]
Finally, note that geodesic curvature $k_g$ and geodesic equations depend only on the metric coefficients $(g_{ij})$, we can also define them for any Riemannian 2-manifold.

In conclusion: any concepts defined on surfaces in $\mathbb{R}^3$ using only $(g_{ij})$ the metric coefficients can be
generalized to Riemannian manifold!

And the geometry related only to the metric coefficients is called the intrinsic geometry of the surface.