1 Normal coordinates

The existence of the normal coordinates is as follows: For each point \( p \in M \), there exists the normal coordinate \( (U, x^1, \ldots, x^m) \) at \( p \), i.e. we have \( x^i(p) = 0 \), \( g_{ij}(p) = \delta_{ij} \) and \( \Gamma^k_{ij}(p) = 0 \) for \( 1 \leq i, j, k \leq m \).

2 Laplacian-Beltrami operator, the musical isomorphism

- **Musical isomorphism:** \( \omega(Y) = g(X,Y) \) gives the map between \( X \) and \( \omega \). Flat \( \flat \) is the map \( X \rightarrow \omega, \# \) is the inverse map, i.e. \( X^\flat(Y) = g(X,Y) \) and \( \omega(Y) = g(\omega^\#, Y) \) for every vector field \( Y \).

- **The divergence operator.** \( div(X) := tr(\nabla X) \) and \( \delta \alpha = -tr(\nabla \alpha) \) (the co-differential of \( \alpha \)) where \( \nabla X \) is the endormorphism \( Y \rightarrow \nabla_Y X \) and the \( \nabla \alpha \) is a covariant two-tensor, where the trace is computed with respect to the Riemannnian metric. In the local frame \( \{e_i\}_{i=1}^n \), write
  \[
  \nabla_{e_i}X = \sum_{j} (\nabla_{e_i}X)^j e_j.
  \]
  then
  \[
  div(X) = \sum_{i=1}^n (\nabla_{e_i}X)^i.
  \]
  To get \( \delta \alpha \), write
  \[
  \nabla_{e_i}\alpha = \sum_{j} (\nabla_{e_i}\alpha)_j e^*_j.
  \]
  Lift the index by taking \( g^{ik} \) to get \( \sum_{j=1}^n (\nabla_{e_i}\alpha)_j g^{ik} \), then take the trace by taking \( k = i \), so we get
  \[
  \delta \alpha = - \sum_{i,j=1}^n g^{ij}(\nabla_{e_i}\alpha)_j.
  \]
  Using the "musical isomorphism", these two notations can be viewed as equivalent (In fact, \( \delta(X^\flat) = -div(X) \). and \( div(\alpha^\#) = -\delta \alpha \).
• The gradient of $f$. Let $f \in C^\infty(M)$, define a tangent vector field $\text{grad}(f) \in \Gamma(TM)$, by

$$g(\text{grad}(f), X) = df(X) = X(f),$$

for every smooth tangent vector field $X$. The tangent vector field $\text{grad}(f)$ is called the gradient of $f$. In terms of local coordinate $(U; x^i)$,

$$\text{grad}(f) = \sum_{j=1}^m \left( \sum_{i=1}^m g^{ij} \frac{\partial f}{\partial x^i} \right) \frac{\partial}{\partial x^j},$$

where $g^{ij} = g(\partial/\partial x^i, \partial/\partial x^j), (g^{ij}) = (g_{ij})^{-1}$.

We have

$$(\text{grad}(f))^\flat = df.$$

• Beltrami-Laplace operator. Let $f \in C^\infty(M)$, define $\triangle f = \text{div} (\text{grad}(f))$. It is called the Beltrami-Laplace operator. The operator $\triangle f = \text{div} \circ \text{grad} : C^\infty(M) \to C^\infty(M)$ is a very important differential operator. From above we We have

$$\triangle f = -\delta df.$$

### 3 Some Formulas

In this notes, we always assume that $M$ is a Riemannian manifold with the Levi-Civita connection $\nabla$, and $X, Y, ...$ are smooth vector fields.

• A basis formula for $d$. Let $(M, g)$ be a Riemannian manifold with the Levi-Civita connection $\nabla$. Let $\{e_i\}$ be a local frame field on $M$ (i.e. a basis for $\Gamma(U, TM)$) and $\{\omega^i\}$ be its dual, i.e. $\omega^j(e_i) = \delta^j_i$. Then, for every smooth differential r-form $\theta$,

$$d\theta = \sum_i \omega^j \wedge \nabla_{e_i} \theta. \quad (1)$$

Proof: First notice that it is independent of the choice of coordinates. So we choose normal coordinates $x^i$, i.e. we have $x^i(p) = 0, g_{ij}(p) = \delta_{ij}$ and $\Gamma^k_{ij}(p) = 0$ for $1 \leq i, j, k \leq m$. Let $\{e_i\}$ is a local orthonormal
frame field on $M$ with $e_i(p) = \frac{\partial}{\partial x_i}|_p$, and let $\{\omega^j\}$ be the dual to $\{e_i\}$. We claim that

$$(\nabla_{e_i} \omega^j)(p) = 0.$$ 

In fact, since $\delta_{jk} = \omega^j(e_k) = (e_k, \omega^j)$, we have

$$d\delta_{jk} = 0 = (\nabla e_k, \omega^j) + (e_k, \nabla \omega^j),$$

i.e. $\nabla \omega^j(e_k) = \omega^j(\nabla e_k)$, hence

$$\nabla_{e_i} \omega^j = -\sum_{k=1}^{m} \Gamma^j_{ik} \omega^k.$$ 

Thus we get, using $\Gamma^j_{ik}(p) = 0$,

$$(\nabla_{e_i} \omega^j)(p) = 0.$$ 

Therefore the claim holds.

Now, since $\nabla_{\partial/\partial x^j} dx^j = 0$, we have, for $\theta = f dx^{i_1} \wedge \cdots \wedge dx^{i_r}$,

$$\sum_i \omega^j \wedge \nabla_{e_i} \theta = \sum_i \frac{\partial f}{\partial x_i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r} = d\theta.$$

**Interior product** For any vector field $X$, $\iota(X)$ sends $r$-form to $r - 1$ defined by, for every $r$-form $\omega$ and vector fields $Y_1, \ldots, Y_{r-1}$,

$$(\iota(X)\omega)(Y_1, \ldots, Y_{r-1}) = \omega(X, Y_1, \ldots, Y_{r-1}).$$

Let $\eta$ be its volume form of $M$. For every smooth tangent vector field $X$,

$$d(\iota(X)\eta) = \text{div}(X)\eta,$$

where $\iota(X)$ is the interior product.

**Proof:** By definition, $\eta = \sqrt{G} dx^1 \wedge \cdots \wedge dx^m$, we claim $\iota(X)\eta = \omega$ where

$$\omega = \sum_{i=1}^{m} (-1)^{i+1} \sqrt{G} X^i dx^1 \wedge \cdots \wedge dx^i \wedge \cdots \wedge dx^m.$$
We now prove it. Indep. of the choice of coordinates. Choose normal coordinates $x^i$. Then, at the point $p$, $\eta = dx^1 \wedge \cdots \wedge dx^m$ and $X = \sum_{j=1}^m X^j e_j$, hence

$$\iota(e_j)(\omega^1 \wedge \cdots \wedge \omega^m) = (-1)^{j+1} \omega^1 \wedge \cdots \hat{\omega}^j \wedge \cdots \wedge \omega^m.$$ 

This proves the claim. The rest of proof follows easily.

- **Divergence theorem:** Let $(M, g)$ be a compact oriented Riemannian manifold, then, for every smooth tangent vector field $X$,

$$\int (\text{div} X) \eta = 0,$$

where $\eta$ is the volume form.

## 4 Hodge-Laplacian Operator

We first extend the codifferential operator $\delta$ to acting on $r$-forms.

- **Global inner product for differential forms:** We first define the inner product for differential forms. Let $\phi, \psi$ are two $r$-forms. Let $(U, x^t)$ be a local coordinate. We write

$$\phi|_U = \frac{1}{r!} \phi_{i_1 \cdots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r},$$

$$\psi|_U = \frac{1}{r!} \psi_{j_1 \cdots j_r} dx^{j_1} \wedge \cdots \wedge dx^{j_r}.$$

We define, the inner product $< , >$ of $\phi, \psi$ as

$$< \phi, \psi > = \frac{1}{r!} \phi^{i_1 \cdots i_r} \psi_{i_1 \cdots i_r} = \sum_{i_1 < \cdots < i_r} \phi^{i_1 \cdots i_r} \psi_{i_1 \cdots i_r},$$

where $\phi^{i_1 \cdots i_r} = g^{i_1 j_1} \cdots g^{i_r j_r} \phi_{j_1 \cdots j_r}$. It is important to note that the definition is independent of the choice of local coordinates. We also have $< \phi, \phi > \geq 0$ and $< \phi, \phi > = 0$ if and only if $\phi = 0$.

We now define the **global** inner product of $\phi, \psi$ as

$$(\phi, \psi) = \int_M < \phi, \psi > \eta,$$

where $\eta$ is the volume form of $M$. 
• **The exterior differential operator** $d$ and its co-operator: Denote by $\Lambda^r(M)$ the set of smooth $r$-forms on $M$. Let $(\ , \ )$ be the (global) inner product defined above. As the formal adjoint operator of the exterior differential operator $d$, the codifferential operator $\delta : \Lambda^{r+1}(M) \to \Lambda^r(M)$ is defined by, for every $\phi \in \Lambda^r(M), \psi \in \Lambda^{r+1}(M)$,

$$(d\phi, \psi) = (\phi, \delta \psi).$$

• **Hodge-star operator** $*$: In order to find the expression of the codifferential operator $\delta$, we introduce the Hodge-star operator $*$, which is an isomorphism $* : \Lambda^r(M) \to \Lambda^{m-r}(M)$ defined by, for every $\phi, \eta \in \Lambda^r(M)$,

$$\phi \wedge (*\psi) = (\phi, \psi) \eta.$$ Let $\omega$ be a $r$-form. Let $(U, x^i)$ be a local coordinate. We write

$$\omega|U = \frac{1}{r!} \sum_{i_1, \ldots, i_r} a_{i_1 \ldots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r}.$$ Then

$$*\omega = \frac{\sqrt{G}}{r!(m-r)!} \delta^{1 \ldots m}_{i_1 \ldots i_r} a^{i_1 \ldots i_r} dx^{i_{r+1}} \wedge \cdots \wedge dx^{i_m},$$

where

$$a^{i_1 \ldots i_r} = g^{i_1 j_1} \cdots g^{i_r j_r} a_{j_1 \ldots j_r},$$

and $\delta^{1 \ldots m}_{i_1 \ldots i_m}$ is the Levi-Civita permutation symbol, i.e. $\delta^{1 \ldots m}_{i_1 \ldots i_m} = 1$ if $(i_1 \ldots i_m)$ is an even permutation of $(1 \ldots m)$, $\delta^{1 \ldots m}_{i_1 \ldots i_m} = -1$ if $(i_1 \ldots i_m)$ is an odd permutation of $(1 \ldots m)$, $\delta^{12 \ldots m}_{1 \ldots m} = 0$ otherwise. It can be shown that $*\omega$ is independent of the choice of local coordinates. So $*\omega$ is a globally defined $(m-r)$-form (it can be regarded as an alternative definition). The operator $*$ which sends $r$-forms to $(m-r)$-forms.

It has the following properties, for any $r$-forms $\phi$ and $\psi$:

1. $\phi \wedge *\psi = (\phi, \psi) \eta$,
2. $*\eta = 1, *1 = \eta$, 

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Expression of $\delta$ in terms of the Hodge-Star operator. Define
\[ \delta = (-1)^{mr+1} \circ d \circ \ast : \Lambda^{r+1}(M) \to \Lambda^r(M), \]
where $\Lambda^r(M)$ is the set of smooth $r$-forms, is called the \textit{codifferential operator}. It is easy to verify that $\delta \circ \delta = 0$. We also have the following very important property for $\delta$: For $\phi \in \Lambda^r(M), \psi \in \Lambda^{r+1}(M)$, we have
\[ (d\phi, \psi) = (\phi, \delta \psi), \]
i.e. $\delta$ is \textit{conjugate} to $d$. So $(-1)^{mr+1} \circ d \circ \ast$ is the expression of the \textit{co-differential operator} $\delta$.

\textit{Proof.} Note
\[
d(\phi \wedge \ast \psi) = d\phi \wedge \ast \psi + (-1)^r \phi \wedge d(\ast \psi)
\]
\[
= d\phi \wedge \ast \psi + (-1)^r(-1)^{mr+r} \phi \wedge (\ast d \ast \psi)
\]
\[
= d\phi \wedge \ast \psi - \phi \wedge \ast \delta \psi.
\]
Then desired identity is obtained by applying the Stokes theorem.

Another expression of the co-differential operator $\delta$: Let $\{e_i\}$ be a local frame field on $M$ compatible with the orientation of $M$. Let $g_{ij} = g(e_i, e_j)$, and $(g^{ij}) = (g_{ij})^{-1}$. Then the codifferential operator $\delta$ can be written as
\[
\delta \alpha = - \sum_{i,j=1}^m g^{ij} (\nabla_{e_i} \alpha), \quad \text{for every} \quad \alpha \in \Lambda^r(M).
\]

If $\{e_i\}$ is orthonormal, then we can write
\[
\delta = - \sum_{j=1}^m \iota(e_j) \nabla_{e_j},
\]
(2)
where $i(X)$ is the interior product operator, i.e. for every $\alpha \in \Lambda^r(M)$, and for every tangent vector fields $X_1, \ldots, X_{r-1}$.

$$\delta\alpha(X_1, \ldots, X_{r-1}) = -\sum_{j=1}^{m} \iota(e_j) (\nabla_{e_j} \alpha)(X_1, \ldots, X_{r-1})$$

$$= -\sum_{j=1}^{m} (\nabla_{e_j} \alpha)(e_j, X_1, \ldots, X_{r-1}).$$

**Proof:** For $p \in M$, choose the normal coordinate $(U, x^1, \ldots, x^m)$ at $p$. Let $\{e_i\}$ is a local orthonormal frame field on $M$ with $e_i(p) = \frac{\partial}{\partial x_i}|_p$, and let $\{\omega^j\}$ be the dual to $\{e_i\}$. Then

$$(\nabla_{e_j} \omega^j)(p) = 0.$$ To prove

$$\delta(\alpha) = -\sum_{j=1}^{m} \iota(e_j) (\nabla_{e_j} \alpha). \quad (3)$$

We need only to verify it at each point $p \in M$. Since the operator is linear, without loss of generality, we assume that

$$\alpha = f \omega^1 \wedge \cdots \wedge \omega^r.$$ Hence

$$\nabla_{e_j} \alpha = \nabla_{e_j} (f \omega^1 \wedge \cdots \wedge \omega^r) + f \nabla_{e_j} (\omega^1 \wedge \cdots \wedge \omega^r)$$

$$= e_j(f) \omega^1 \wedge \cdots \wedge \omega^r + f \nabla_{e_j} (\omega^1 \wedge \cdots \wedge \omega^r).$$

Using $(\nabla_{e_j} \omega^j)(p) = 0$, (only) at the point $p$, we have

$$\nabla_{e_j} \alpha = e_j(f) \omega^1 \wedge \cdots \wedge \omega^r.$$ Hence, at the point $p$, we have

$$\iota(e_j)(\nabla_{e_j} \alpha) = e_j(f)(\iota(e_j)(\omega^1 \wedge \cdots \wedge \omega^r)).$$

Because

$$\iota(e_j)(\omega^1 \wedge \cdots \wedge \omega^r) = (-1)^{j+1} \omega^1 \wedge \cdots \wedge \hat{\omega}^j \wedge \cdots \wedge \omega^r,$$
we have
\[ \iota(e_j)(\nabla e_j \alpha) = (-1)^{j+1} e_j(f) \omega^1 \wedge \cdots \wedge \hat{\omega}^j \wedge \cdots \wedge \omega^r. \]

This tells us, at the point \( p \), that
\[ -\sum_{j=1}^{m} \iota(e_j)(\nabla e_j \alpha) = -\sum_{j} (-1)^{j+1} e_j(f) \omega^1 \wedge \cdots \wedge \hat{\omega}^j \wedge \cdots \wedge \omega^r. \quad (4) \]

We now calculate the left-hand side. By defintion, \( \delta = (-1)^{n(r+1)+1} \star d \star \)
We have,
\[ \delta(\alpha) = (-1)^{n(r+1)+1} \star d \star (\alpha) = (-1)^{n(r+1)+1} \star d \star (f \omega^1 \wedge \cdots \wedge \omega^r) \]
\[ = (-1)^{n(r+1)+1} \star d(f \omega^{r+1} \wedge \cdots \wedge \omega^m) \]
\[ = (-1)^{n(r+1)+1} \star (\sum_j e_j(f) \omega^j \wedge \omega^{r+1} \wedge \cdots \wedge \omega^m) \]

Note that
\[ \star (\omega^j \wedge \omega^{r+1} \wedge \cdots \wedge \omega^m) = (-1)^{(r-1)(n-r-1)+(r-j)} \omega^1 \wedge \cdots \wedge \hat{\omega}^j \wedge \cdots \wedge \omega^r, \]
hence
\[ \delta(\alpha) = \sum_j (-1)^{n(r+1)+1} (-1)^{(r-1)(n-r-1)+(r-j)} e_j(f) \omega^1 \wedge \cdots \wedge \hat{\omega}^j \wedge \cdots \wedge \omega^r \]
\[ = \sum_j (-1)^j e_j(f) \omega^1 \wedge \cdots \wedge \hat{\omega}^j \wedge \cdots \wedge \omega^r. \]

Comparing the above identity with (4), we conclude that (2) holds at every point \( p \). Hence the theorem holds.

- **Hodge-Laplace operator.** We define the Hodge-Laplace operator
  \[ \tilde{\Delta} = d \delta + \delta d : \Lambda^r(M) \to \Lambda^r(M). \]

For \( f \in C^\infty(M) \), then \( \delta(f) = 0 \), so
\[ \tilde{\Delta}(f) = \delta(df) = -\star d \star df, \quad \tilde{\Delta} f \eta = \star \tilde{\Delta} f = -d \star df. \]

Let \((U, x^i)\) be a local coordinate, then
\[ df|_U = \frac{\partial f}{\partial x^i} dx^i, \]
\[
*df|_U = \frac{\sqrt{G}}{(m-1)!} \delta_{1...m}^{i_1...i_m} g^{i_1j} \frac{\partial f}{\partial x^j} dx^{i_2} \wedge \cdots \wedge dx^{i_m}
\]

\[
= \sqrt{G} \sum_{i=1}^{m} (-1)^{i+1} g^{i_2j} \frac{\partial f}{\partial x^j} dx^1 \wedge \cdots \wedge dx^i \wedge \cdots \wedge dx^m.
\]

Hence

\[
(\tilde{\Delta} f)|_U = -d(*df)|_U = -\frac{\partial}{\partial x^i} \left( \sqrt{G} g^{i_2j} \frac{\partial f}{\partial x^j} \right) dx^1 \wedge \cdots \wedge dx^m
\]

This tells us

\[
\tilde{\Delta} f = -\Delta f.
\]

So \(-\tilde{\Delta}\) when acts on \(C^\infty(M)\) is the Beltrami-Laplace operator \(\Delta\).

5 Bochner-Weitzenbock Formulas

The Bochner-Weitzenbock Formulas, sometimes referred to as the Bochner technique, is one of the most important technique in the theory of geometric analysis.

**Bochner Formula:** For \(f \in C^3(M)\),

\[
\frac{1}{2} \Delta |\text{grad} f|^2 = |\text{Hess} f|^2 + \langle \text{grad} f, \text{grad}(\Delta f) \rangle + \text{Ric}(\text{grad} f, \text{grad} f),
\]

where \(\text{grad} f\) is the gradient of \(f\) which is given by \(\langle \text{grad} f, Y \rangle = Y(f)\), the \(\text{Hess}(f)\) is the two tensor \(\nabla(\nabla f)\), which is defined by for two vector fields \(X, Y\),

\[
\text{H}(f)(X, Y) = \langle \nabla_X (\text{grad} f), Y \rangle
\]

and the Laplacian-Beltrami operator is \(\Delta f = \text{tr}(\text{Hess}(f))\).

**Proof:** Fix \(p \in M\), let \(\{X_i\}_{i=1}^m\) be a local o.n. frames such that \(\langle X_i, X_j \rangle = \delta_{ij}, \nabla_{X_i} X_j(p) = 0\). Computation at \(p\) gives

\[
\frac{1}{2} \Delta |\text{grad} f|^2 = \frac{1}{2} X_i X_i \langle \text{grad} f, \text{grad} f \rangle = \sum_i X_i \langle \nabla_{X_i} \text{grad} f, \text{grad} f \rangle
\]
\[ \sum_i X_i \text{Hess}(f)(X_i, \text{grad} f) = \sum_i X_i \text{Hess}(f)(\text{grad} f, X_i) \quad \text{(Hessian is symmetric)} \]

\[ = \sum_i X_i < \nabla_{\text{grad} f}(\text{grad} f), X_i > \]

\[ = \sum_i < \nabla X_i \nabla_{\text{grad} f}(\text{grad} f), X_i > + < \nabla_{\text{grad} f}(\text{grad} f), \nabla X_i > \]

\[ = \sum_i < \nabla X_i \nabla_{\text{grad} f}(\text{grad} f), X_i > \quad \text{the other term vanishes at } p \]

\[ = \sum_i < R(X_i, \text{grad} f \text{grad} f), X_i > + \sum_i < \nabla_{\text{grad} f} \nabla X_i (\text{grad} f), X_i > \]

\[ + \sum_i < \nabla_{[X_i, \text{grad} f]} \text{grad} f, X_i > . \]

The first term is by definition \( \text{Ric}(\text{grad} f, \text{grad} f) \); the second term is

\[ \sum_i (\text{grad} f) < \nabla X_i \text{grad} f, X_i > - < \nabla X_i \text{grad} f, \nabla_{\text{grad} f} \nabla X_i > \]

\[ = (\text{grad} f) \sum_i < \nabla X_i \text{grad} f, X_i > - 0 \; \text{at} \; p \]

\[ = (\text{grad} f) (\Delta f) < \text{grad} f, \text{grad} (\Delta f) > \]

and the third term is

\[ \sum_i \text{Hess}(f)([X_i, \text{grad} f], X_i) = \sum_i \text{Hess}(f)(\nabla X_i \text{grad} f - \nabla_{\text{grad} f} X_i, X_i) \]

\[ = \sum_i \text{Hess}(f)(\nabla X_i \text{grad} f, X_i) - \text{Hess}(f)(\nabla_{\text{grad} f} X_i, X_i) \]

\[ = \sum_i \text{Hess}(f)(\nabla X_i \text{grad} f, X_i) - 0 \; \text{at} \; p \]

\[ = \sum_i \text{Hess}(f)(X_i, \nabla X_i \text{grad} f) \]

\[ = \sum_i < \nabla X_i \text{grad} f, \nabla X_i \text{grad} f > = |\text{Hess}(f)|^2. \]

The theorem follows.

We now look for the differential form version of the Bochner Formula. The goal is to prove the following theorem:

**Bochner’s formula.** Let \( e_1, \ldots, e_m \) be a local orthonormal frame field, with the dual frame field \( \omega^1, \cdots, \omega^m \). Let \( \eta \) be a \( r \)-form on \( M \). Then

\[
\frac{1}{2} \Delta |\eta|^2 = <\nabla \eta, \eta> - |\nabla \eta|^2 + <\omega^i \wedge \iota(e_j) R(e_i, e_j) \eta, \eta>.
\]
Remark: The above $\Delta$ is the Laplacian-Hodge operator, which is different (which restricting to the functions) from the Laplacian-Beltrem operator by a negative sign.

- The Weitzenbock formula above the expression of $\Delta$: We first consider the function case. For functions, i.e. 0-form $f$, by definition, $g(\text{grad}(f), Y) = Y(f)$ and $df(Y) = Y(f)$, so $\text{grad}(f)\flat = df$. Hence, from above, $\delta(df) = -\text{div}(\text{grad}(f))$. Thus

$$\Delta f = \delta df = -\text{div}(\text{grad}(f)) = -tr \nabla^2 f,$$

where $tr \nabla^2 := \sum_{i=1}^{m} \nabla e_i \nabla e_i$ for an orthonormal basis $\{e_i\}$. Hence, we have the Weitzenbock formula for functions:

$$\Delta f = -tr \nabla^2 f.$$

In general, let $\{e_i\}$ be a local frame for a Riemannian manifold $(M, g)$, define

$$tr \nabla^2 : \Lambda^r(M) \to \Lambda^r(M)$$

as

$$tr \nabla^2 (\alpha) = g^{ij}(\nabla e_i \nabla e_j - \nabla \nabla e_i e_j )\alpha,$$

for every $\alpha \in \Lambda^r(M)$.

Weitzenbock formula for 1-forms: For 1-form $\alpha$, we have

$$\Delta \alpha(X) = -tr \nabla^2 \alpha(X) + r(\alpha^#, X),$$

where $r$ is the Ricci tensor of $(M, g)$.

Proof. The right hand side is independent of the choice of our orthonormal frame field. Therefore, we only need to verify it at every point $p \in M$. To do so, we choose normal coordinates centered at $p$ and put at $p$,

$$e_i = \frac{\partial}{\partial x^i}.$$
Let $\omega^j$ be its dual frame. Then, always at $p$,

$$\nabla e_i e_j = 0.$$  

This also gives, at $p$,

$$\nabla e_j \omega^j = 0.$$  

Using (1) and (2), we then have at $p$,

$$\begin{align*}
(\delta d\alpha)(X) &= (\delta(\omega^j \wedge \nabla e_j \alpha))(X) \\
&= -((\nabla e_i (\omega^j \wedge \nabla e_j \alpha))(e_i, X) \\
&= -(\omega^j \wedge \nabla e_i \nabla e_j \alpha)(e_i, X) \\
&= -\nabla e_i \nabla e_j \alpha(X) + X^j \nabla e_i \nabla e_j \alpha(e_i).
\end{align*}$$

$$\begin{align*}
(d\delta \alpha)(X) &= (\omega^j \nabla e_j (\delta \alpha))(X) \\
&= X^j \nabla e_j (\delta \alpha) \\
&= -X^j \nabla e_j \nabla e_i \alpha(e_i).
\end{align*}$$

Hence,

$$\begin{align*}
\Delta \alpha(X) &= -tr\nabla^2 \alpha(X) + X^j \nabla e_i \nabla e_j \alpha(e_i) - X^j \nabla e_j \nabla e_i \alpha(e_i) = X^j R(e_i, e_j) \alpha(e_i),
\end{align*}$$

where where $X = X^k e_k$ and

$$R(e_i, e_j) \alpha = (\nabla e_i \nabla e_j - \nabla e_j \nabla e_i) \alpha.$$  

We now claim that

$$X^j R(e_i, e_j) \alpha(e_i) = r(\alpha^#, X).$$

In fact, write $\alpha = \alpha_k \omega^k$, then $\alpha^# = \alpha_k e_k$, and

$$X^j R(e_i, e_j) \alpha(e_i) = X^j \alpha_k R(e_i, e_j) \omega^k(e_i)$$

$$= X^j \alpha_k R(e_i, e_j) \omega^k(e_i)$$

$$= -X^j \alpha_k R(kmji) \omega^m(e_i)$$

$$= -X^j \alpha_k R_{klij}$$

$$= -r(\alpha^#, X).$$
This proves the statement.

We now derive the formula for a general $r$-form. For the purpose, we define the second covariant derivative as

$$\nabla^2_{XY} = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}.$$ 

**Weitzenbock’s formula.** Let $e_1, \ldots, e_m$ be a local orthonormal frame field, with the dual frame field $\omega^1, \ldots, \omega^m$. Then the Hodge-Laplace operator $\Delta$ acting on $r$-differential forms is given by

$$\Delta = -\sum_{i=1}^m \nabla^2_{e_i e_i} - \sum_{i,j} \omega^i \wedge \iota(e_j) R(e_i, e_j).$$

**Proof.** The right hand side is independent of the choice of our orthonormal frame field. Therefore, we only need to verify it at every point $p \in M$. To do so, we choose normal coordinates centered at $p$ and put at $p$,

$$e_i = \frac{\partial}{\partial x^i}.$$ 

Then, always at $p$,

$$\nabla_{e_i} e_j = 0.$$ 

Hence,

$$\nabla^2_{e_i e_i} = \nabla_{e_i} \nabla_{e_i},$$

and also

$$[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0.$$ 

Therefore

$$R(e_i, e_j) = \nabla_{e_i} \nabla_{e_j} - \nabla_{e_j} \nabla_{e_i}.$$ 

Using (1) and (2), we then have at $p$,

$$\delta d = -\iota(e_j) \nabla_{e_j} (\omega^i \wedge \nabla_{e_i})$$

$$= -\iota(e_j) (\omega^i \wedge \nabla_{e_j} \nabla_{e_i})$$

$$= -\nabla_{e_j} \nabla_{e_j} \omega^i + \omega^i \wedge \iota(e_j) \nabla_{e_j} \nabla_{e_i}.$$
To calculate $d\delta$, we note that, since $\nabla_{e_i}\omega^j = 0$,

$$\iota(e_j)\nabla_{e_i} = \nabla_{e_i}\iota(e_j).$$

Hence,

$$d\delta = -\omega^i \wedge \nabla_{e_i}(\iota(e_j)\nabla_{e_j}) = -\omega^i \wedge \iota(e_j) \nabla_{e_i} \nabla_{e_j}.$$ 

So

$$\triangle = d\delta + \delta d = -\sum_{i=1}^{m} \nabla^2_{e_i e_i} - \sum_{i,j} \omega^i \wedge \iota(e_j) R(e_i, e_j).$$

This proves the statement.

**Remark:** On functions, i.e. 0-form $f$, we have

$$R(e_i, e_j)f = f R(e_i, e_j)1 = 0.$$ 

Hence, we have, for a local orthonormal frame field,

$$\triangle f = -\sum_{i=1}^{m} \nabla^2_{e_i e_i} f.$$ 

• **Bochner’s formula:** We first prove the following theorem.

**Theorem.** For any smooth differential form $\eta$ ($r$-forms),

$$-\triangle |\eta|^2 = 2|\nabla \eta|^2 + 2 < \sum_i \nabla^2_{e_i e_i} \eta, \eta >,$$

where

$$|\nabla \eta|^2 = \sum_i |\nabla_{e_i} \eta|^2.$$

Remark: Consider the Euclidean case $\mathbb{R}^2$ and $\eta = f$. Then

$$-\triangle |f|^2 = 2f_x f_x + 2f_y f_y + 2f_{xx} f + 2f_{yy} f = 2|\nabla f|^2 + 2 < f_{xx} + f_{yy}, f >.$$
The theorem is motivated by it, and can be derived by direct computation. We only verify it for 1-form $\eta$. We only need to verify it at every point $p \in M$. Let $\{e_1, \ldots, e_m\}$ be a local orthonormal frame fields and $\{\omega_1, \ldots, \omega_m\}$ be the coframe fields. Write

$$\eta = \sum_i \eta_i \omega^i,$$

then, at point $p$, using the formula of $\triangle$ for smooth functions,

$$-\frac{1}{2} \triangle |\eta|^2 = \sum_i <\nabla_{e_i} \nabla_{e_i} \eta, \eta > + \sum_i |\nabla_{e_i} \eta|^2.$$

This finishes the proof.

Combining the above theorem with the Weitzenbock’s formula for r-forms, we get

**Bochner’s formula.** Let $e_1, \ldots, e_m$ be a local orthonormal frame field, with the dual frame field $\omega^1, \ldots, \omega^m$. Let $\eta$ be a r-form on $M$. Then

$$\frac{1}{2} \triangle |\eta|^2 = <\triangle \eta, \eta > - |\nabla \eta|^2 + <\omega^i \wedge \iota(e_j) R(e_i, e_j) \eta, \eta >.$$

**Proof:** Choose normal coordinate. From above,

$$\frac{1}{2} \triangle |\eta|^2 = -|\nabla \eta|^2 - \sum_i <\nabla^2_{e_i} \eta, \eta >.$$

And from the Weitzenbock’s formula,

$$\Delta \eta = -\sum_{i=1}^{m} \nabla^2_{e_i} \eta - \sum_{i,j} \omega^i \wedge \iota(e_j) R(e_i, e_j) \eta.$$

Hence,

$$-\sum_{i=1}^{m} \nabla^2_{e_i} \eta = \Delta(\eta) + \sum_{i,j} \omega^i \wedge \iota(e_j) R(e_i, e_j) \eta.$$
Sbustituting it into above, we get

\[ \frac{1}{2} \Delta |\eta|^2 = <\Delta \eta, \eta> - |\nabla \eta|^2 + <\omega^i \wedge \iota(e_j)R(e_i, e_j) \eta, \eta>. \]

For any smooth one-form, we get

**Corollary** Let \( e_1, \ldots, e_m \) be a local orthonormal frame field, with the dual frame field \( \omega^1, \ldots, \omega^m \). Let \( \eta \) be a smooth one-form, then

\[ \frac{1}{2} \Delta |\eta|^2 = <\Delta \eta, \eta> - |\nabla \eta|^2 - r(\eta, \eta), \]

where \( |\nabla \eta|^2 := \sum_i <\nabla_{e_i} \eta, \nabla_{e_i} \eta> \), and writing \( \eta = \sum f_i \omega^i \),

\[ r(\eta, \eta) := \sum_{i,j} r(f_i e_i, f_j e_j) = \sum_{i,j} f_i f_j r(e_i, e_j). \]

**Proof:** We only need to compute, for 1-form \( \eta \),

\[ <\eta, \omega^i \wedge \iota(e_j)R(e_i, e_j) \eta> = <f_i \omega^i, \omega^j \wedge \iota(e_j)R(e_i, e_j) f_k \omega^k> \]
\[ = -f_i f_k <\omega^i, \omega^j \wedge \iota(e_j)R(e_i, e_j) \omega^k> \]
\[ = -f_i f_k <\omega^i, \omega^j \wedge \iota(e_j)R_{k\langle i\rangle} \omega^m> \]
\[ = -f_i f_k <\omega^i, R_{k\langle i\rangle j} \omega^j> \]
\[ = -f_i f_k R_{k\langle i\rangle j} \]
\[ = -f_i f_k R_{kl} \]
\[ = -r(\eta, \eta). \]

**Theorem (Bochner)** Let \( M \) be a compact Riemannian manifold. If \( M \) has positive Ricci curvature, then \( M \) has no nontrivial harmonic 1-form, thus,

\[ H^1_{\text{dR}}(M, \mathbb{R}) = \{0\}. \]
Proof. We integrate the formula above, and using the divergence theorem,

\[ 0 = -\int_M \Delta |\omega| \eta = 2 \int_M (|\nabla \omega|^2 + r(\omega, \omega)) \eta. \]

By the assumption, the integrand on the right hand side is pointwise nonnegative. It therefore has to vanish identically. Hence \( r(\omega, \omega) = 0 \), which implies that \( \omega = 0 \) since the Ricci curvature on \( M \) is positive.

This proves the statement.

6 Proof of Garding’s inequality

Theorem Garding’s inequality: There exist constant \( c_1, c_2 > 0 \), such that for every \( f \in \Lambda^*(M) \), we have

\[ (\Delta f, f) \geq c_1 \|f\|^2_1 - c_2 \|f\|^2_0. \]

Proof. For every \( f \in \Lambda^*(M) \), from Bochner’s formula,

\[ < \Delta f, f > = \frac{1}{2} \Delta |f|^2 + |\nabla f|^2 - < \omega^i \wedge \iota(e_j) R(e_i, e_j) f, f > \geq \frac{1}{2} \Delta |f|^2 + |\nabla f|^2 - a_1 |f|^2, \]

where \( a_1 \) is a constant independent of \( f \). Note that the last inequality holds because \( < \omega^i \wedge \iota(e_j) R(e_i, e_j) f, f > \) does not depend on the derivative(differential) of \( f \) (Note: although \( R(e_i, e_j) \) depends on the derivative(differential), but since \( R(e_i, e_j)(\alpha f) = \alpha R(e_i, e_j) f \) for every \( \alpha \in C^\infty(M) \), so \( < \omega^i \wedge \iota(e_j) R(e_i, e_j) f, f > \) is a quadratic form on \( \Lambda^*(M) \), its coefficients only depend on \( M \), and since \( M \) is compact, such constant \( a_1 \) exists). Taking the integration on \( M \) and by definition, we get

\[ (\Delta f, f) \geq c_1 \|f\|^2_1 - c_2 \|f\|^2_0 + \frac{1}{2} \int_M \Delta |f|^2 \eta. \]

But by Stokes theorem,

\[ \int_M \Delta |f|^2 \eta = 0. \]

This proves the Garding’s inequality.
7 Hodge Theory

In this section, we denote the Hodge-Laplace operator by $\triangle$. Let $\mathcal{H}^r(M) = \ker \triangle$ and $\mathcal{H} = \bigoplus \mathcal{H}^r(M)$. Let $\Lambda^r(M) = \bigoplus_{r=0}^{\infty} \Lambda^r(M)$.

The Hodge theorem Let $(M, g)$ be an $n$-dimensional compact oriented Riemannian manifold without boundary. For each integer $0 \leq r \leq n$, $\mathcal{H}^r(M)$ is finite dimensional, and there exists a bounded linear operator $G : \Lambda^r(M) \to \Lambda^r(M)$ (called Green’s operator) such that
(a) $\ker G = \mathcal{H}$;
(b) $G$ keeps types, and commute with the operators $\ast, d$ and $\delta$;
(c) $G$ is a compact operator, i.e. the closure of image of an arbitrary bounded subset of $\Lambda^r(M)$ under $G$ is compact;
(d) $I = \mathcal{H} + \triangle \circ G$, where $I$ is the identity operator, and $\mathcal{H}$ is the orthogonal projection from $\Lambda^r(M)$ to $\mathcal{H}$ with respect to the inner product $(\ , \ )$.

From the Hodge theorem, since $I = \mathcal{H} + \triangle \circ G$, we can write (called the Hodge-decomposition)

Corollary (Hodge-decomposition)

$$\Lambda^r(M) = \triangle (\Lambda^r(M)) \oplus \mathcal{H}^r(M)$$

$$= d\delta \Lambda^r(M) \oplus \delta d \Lambda^r(M) \oplus \mathcal{H}^r(M)$$

$$= d\Lambda^{r-1}(M) \oplus \delta \Lambda^{r+1}(M) \oplus \mathcal{H}^r(M).$$

To prove this theorem, basically we need to show two things: (1): $\mathcal{H}$ is a finite dimensional vector space, (2): Write $\Lambda^r(M) = \mathcal{H} \oplus \mathcal{H}^\perp$, where $\mathcal{H}^\perp$ is the orthogonal complement of $\mathcal{H}$ with respect to $(\ , \ )$, we need to show that $\triangle : \mathcal{H}^\perp \to \mathcal{H}^\perp$ and $\triangle$ is one-to-one and onto. (note that: for every $\phi \in \Lambda^r(M)$, $\psi \in \mathcal{H}$, $(\triangle \phi, \psi) = (\phi, \triangle \psi) = 0$, so $\triangle \phi \in \mathcal{H}^\perp$. Hence $\triangle : \mathcal{H}^\perp \to \mathcal{H}^\perp$). Once (1) and (2) are proved, then we take $G|_{\mathcal{H}} = 0$, and $G|_{\mathcal{H}^\perp} = \triangle^{-1}$. This will prove the Hodge theorem.

To do so, we first note that the operator $\triangle$ is positive (i.e. its eigenvalues are all positive). In fact, write $P = d + \delta$. Then it is easy to verify that both $P$ are $\triangle$ are self-dual, and $\triangle = P^2$. Hence

$$(\triangle \phi, \phi) = (P \phi, P \phi) = (d \phi, d \phi) + (\delta \phi, \delta \phi) \geq 0.$$
So $\triangle$ is an elliptic self-adjoint operator. We therefore use the “theory of elliptic (self-adjoint) differential operator”. To do so, we need first introduce the concept of “Sobolov space”.

Let $s$ be a nonnegative integer. Define the inner product $(\ , \ )_s$ on $\bigwedge^*(M)$ as follows: for every $f_1, f_2 \in \bigwedge^*(M)$, define
\[
(f_1, f_2)_s = \sum_{k=0}^{s} \int_M <\nabla^k f_1, \nabla^k f_2 > \ast 1,
\]
where $\ast 1$ is the volume form on $M$. Let $H_s(M)$ be the completion of $\bigwedge^*(M)$ with respect to the Sobolov norm $\| \cdot \|_s$, which is called the ‘Sobolov space.

We use the following three facts (proofs are omitted):

- **Garding’s inequality:** There exist constant $c_1, c_2 > 0$, such that for every $f \in \bigwedge^*(M)$, we have
  \[
  (\triangle f, f) \geq c_1 \| f \|_1^2 - c_2 \| f \|_0^2.
  \]

**Remark:** This is a variant of so-called *Bocher technique*.

To state the second fact, we introduce the concept of weak derivative: Write $P = d + \delta$ and $\triangle = P^2$. For $\phi \in H_s(M)$ and $\psi \in H_t(M)$, we say $P\phi = \psi$(weak), if for every test form $f \in \bigwedge^*(M)$, we have $(\phi, Pf) = (\psi, f)$. In similar way, $\triangle \phi = \psi$(weak) is defined. If $\phi \in H_s(M)$, $\psi \in H_t(M)$, and $P\phi = \psi$(weak), we denote it by $P\phi \in H_t(M)$.

- **Regularity of the operator $P$:** If $\phi \in H_0(M)$ and $P\phi \in \bigwedge^*(M)$, then $\phi \in \bigwedge^*(M)$.

- **Rellich Lemma:** If $\{\phi_i\} \subset \bigwedge^*(M)$ is bounded in the $\| \cdot \|_1$, then it has a Cauchy subsequence with respect to the norm $\| \cdot \|_0$.

The above theorem about the **Regularity of the operator $P$** implies the following lemma
• The weak form of the Wyle lemma: If $\phi \in H_1(M)$, and $\Delta \phi = \psi$ (weak) with $\psi \in \Lambda^*(M)$, then $\phi \in \Lambda^*(M)$.

Proof of the Hodge Theorem. We first prove that $\mathcal{H}$ is a finite dimensional vector space. If not, there exists an infinite orthonormal set \{\omega_1, \ldots, \omega_n, \ldots\}. By Garding’s inequality, there exist constants $c_1, c_2$ such that for all $i$, we have

$$\|\omega_i\|_1^2 \leq \frac{1}{c_1}\{(\Delta \omega_i, \omega_i) + c_2\|\omega_i\|_0^3\} = \frac{c_2}{c_1}.$$ \hspace{1cm} \text{(1)}

By Rellich Lemma, \{\omega_i\} must have a Cauchy subsequence with respect to the norm $\|\|_0$, which is impossible, since $\|\omega_i - \omega_j\|_0^2 = 2$ for $i \neq j$. This proves that $\mathcal{H}$ is a finite dimensional vector space.

Next, write

$$\bigwedge^\ast(M) = \mathcal{H} \oplus \mathcal{H}^\perp,$$

where $\mathcal{H}^\perp$ is the orthogonal complement of $\mathcal{H}$ with respect to $(\ , \ )$. We now prove a simpler version of Garding’s inequality:

**Garding’s Lemma** there exists a positive constant $c_0$ such that for all $f \in \mathcal{H}^\perp$, we have

$$\|f\|_1^2 \leq c_0(\Delta f, f).$$

*Proof.* If not, there exists a sequence $f_i \in \mathcal{H}^\perp$ with $\|f_i\|_1 = 1$ and $(\Delta f_i, f_i) \to 0$. From Rellich lemma, we assume, WLOG, that $f_i$ is convergent with respect to $\|\|_0$, i.e. there exists $F \in H_0(M)$ such that $\lim_{i \to +\infty} \|F - f_i\|_0 = 0$. We claim that $F = 0$. In fact, from above, 

$$(\Delta f_i, f_i) = \|P f_i\|_0^2 \to 0,$$

hence for every $\phi \in \Lambda^*(M)$,

$$(F, P\phi) = \lim_{i \to +\infty} (f_i, P\phi) = \lim_{i \to +\infty} (P f_i - \phi) = 0.$$

Hence $PF = 0$ (weak). From the regularity of $P$, we have $F \in \Lambda^*(M)$. Hence

$$\Delta F = P(PF) = 0,$$

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so \( F \in \mathcal{H} \). Also, since \( f_i \in \mathcal{H}^\perp \), we have, for every \( \phi \in \mathcal{H} \),
\[
(F, \phi) = \lim_{i \to +\infty} (f_i, \phi) = 0,
\]
so \( F \in \mathcal{H}^\perp \). Thus \( F \in \mathcal{H} \cap \mathcal{H}^\perp \). This implies that \( F = 0 \). This means that \( \lim_{i \to +\infty} \|f_i\|_0 = 0 \). Now, by the Garding inequality, there exist constant \( c_1, c_2 > 0 \) such that
\[
(\Delta f_i, f_i) \geq c_1 \|f_i\|_1^2 - c_2 \|f_i\|_0^2.
\]
Because, from above, both \((\Delta f_i, f_i)\) and \(\|f_i\|_0^2\) converge to zero, so \(\lim_{i \to +\infty} \|f_i\|_1 = 0\), which contradicts the assumption that \(\|f_i\|_1 = 1\). This proves Garding’s lemma.

We now prove that \( \Delta : \mathcal{H}^\perp \to \mathcal{H}^\perp \) and \( \Delta \) is one-to-one and onto.

First we show that \( \Delta : \mathcal{H}^\perp \subset \mathcal{H}^\perp \). In fact, for every \( \phi \in \Lambda^*(M), \psi \in \mathcal{H} \),
\[
(\Delta \phi, \psi) = (\phi, \Delta \psi) = 0,
\]
so \( \Delta \phi \in \mathcal{H}^\perp \). To show \( \Delta \) is one-to-one, let \( \phi_1, \phi_2 \in \mathcal{H}^\perp \), and assume that \( \Delta \phi_1 = \Delta \phi_2 \). Then, from one hand, \( \phi_1 - \phi_2 \in \mathcal{H}^\perp \). On the other hand, since \( \Delta(\phi_1 - \phi_2) = 0 \), \( \phi_1 - \phi_2 \in \mathcal{H} \). Hence \( \phi_1 = \phi_2 \). It remains to show that \( \Delta \) is onto, i.e. for every \( f \in \mathcal{H}^\perp \), there exists \( \phi \in \mathcal{H}^\perp \) such that \( \Delta \phi = f \). This gets down to solve the differential equation \( \Delta \phi = f \) (with unknown \( \phi \)). Let \( B \) be the closure of \( \mathcal{H}^\perp \) in \( H_1(M) \).

From Wyle’s theorem, we only need to solve \( \Delta \phi = f \) in the weak sense, i.e. there exists \( \phi \in B \) such that, for every \( g \in \Lambda^*(M) \),
\[
(\phi, \Delta g) = (f, g).
\]
Since \( \Lambda^*(M) = \mathcal{H} \oplus \mathcal{H}^\perp \), we can write \( g = g_1 + g_2 \) where \( g_1 \in \mathcal{H}, g_2 \in \mathcal{H}^\perp \). So the above identity is equivalent to every \( g_2 \in \mathcal{H}^\perp \),
\[
(\phi, \Delta g_2) = (f, g_2).
\]
So the proof is reduced to the following statement: for every \( f \in \mathcal{H}^\perp \), there exists \( \phi \in B \) such that, for every \( g \in \mathcal{H}^\perp \),
\[
(\phi, \Delta g) = (f, g).
\]

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We now use the **Riesz representation** theorem to prove this statement. In fact, for every $\phi, \psi \in \mathcal{H}^\perp$, define $\langle \phi, \psi \rangle = (\phi, \Delta \psi)$, and consider the linear transformation $L : B \to \mathbb{R}$ defined by $l(g) = (f, g)$ for every $g \in B$. Our goal is to show that we can extend $\langle \ , \ \rangle$ to $B$ such that $l$ is continuous with respect to $\langle \ , \ \rangle$ (or bounded). Then by **Riesz representation** theorem, there exists $\phi \in B$ such that, for every $g \in B$ (in particular for $g \in \mathcal{H}^\perp$),

$$l(g) = \langle \phi, g \rangle.$$ 

This will prove our statement. To extend $\langle \ , \ \rangle$, we compare $\langle \ , \ \rangle$ with $(\ , \ )_1$. From definition, $\langle \ , \ \rangle$ is bilinear. From Garding’s inequality, for every $\phi \in \mathcal{H}^\perp$,

$$\langle \phi, \phi \rangle = (\phi, \Delta \phi) \geq \frac{1}{c_0} \|\phi\|_1^2.$$ 

On the other hand,

$$\langle \phi, \phi \rangle = (\phi, \Delta \phi) = \|P\phi\|_0.$$ 

By direct verification, we have, for every $\phi \in \Lambda^*(M)$,

$$\|P\phi\|_0^2 \leq c \|\phi\|_1^2.$$ 

Hence

$$\langle \phi, \phi \rangle \leq c\|\phi\|_1^2.$$ 

So $\langle \ , \ \rangle$ and $(\ , \ )_1$ are equivalent on $\mathcal{H}^\perp$. So there exists an unique continuation on $B$, and for every $g \in B$, we have

$$\langle g, g \rangle \geq \frac{1}{c_0} \|g\|_1^2.$$ 

To show that $l$ is continuous with respect to $\langle \ , \ \rangle$(or bounded), we notice that

$$|l(g)| = |(f, g)| \leq \|f\|_0 \|g\|_0 \leq \|f\|_0 \|g\|_1 \leq \sqrt{c_0} \|f\|_0 \sqrt{\langle g, g \rangle}.$$ 

So the claim is proved. This finishes the proof that $\Delta$ is onto.

To prove Hodge’s theorem, since, from above, $\Delta : \mathcal{H}^\perp \to \mathcal{H}^\perp$ is one-to-one and onto, we let $G : \Lambda^*(M) \to \Lambda^*(M)$ be defined as follows: $G|_{\mathcal{H}} =$
0, and $G|_{\mathcal{H}} = \triangle^{-1}$. Then we see that $\ker G = \mathcal{H}$ and $I = \mathcal{H} + \triangle \circ G$. The rest of properties are also easy to verify.

This finishes the proof.

- **Application of the Hodge Theory.** Let $M$ be a compact manifold. Denote by $\Lambda^r(M)$ the set of all $r$-forms on $M$. Clearly $\Lambda^0(M)$ is the set of all differential functions on $M$. By the rule of the exterior multiplication, we see that $0 \leq r \leq n$.

The exterior differential operator is a map $d : \Lambda^r(M) \rightarrow \Lambda^{r+1}(M)$, which satisfies conditions:

(i) $d$ is $\mathbb{R}$-linear;

(ii) For $f \in \Lambda^0(M)$, $df$ is the usual differential of $f$, and $d(df) = 0$;

(iii) $d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^r \phi \wedge d\psi$ for any $\phi \in \Lambda^r(M)$ and any $\psi$.

There are three important properties for $d$: (a) $d^2 = 0$ (called the Poincare lemma), (b) For $\omega \in \Lambda^1(M)$ and $X,Y \in \Gamma(TM)$, we have

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]).$$

(c) If $F : M \rightarrow N$, then $F^* \circ d = d \circ F^*$.

A differential $r$-form $\phi \in \Lambda^r(M)$ is said to be **closed** if $d\phi = 0$, and $\phi \in \Lambda^r(M)$ is said to be **exact** if there exists $\eta \in \Lambda^{r-1}(M)$ such that $\phi = d\eta$. Since $d \circ d = 0$, we know that every exact form is also closed. Let $Z^r(M, \mathbb{R})$ denote the set of all (smooth) closed $r$-forms on $M$, and let $B^r(M, \mathbb{R})$ denote the set of all (smooth) exact $r$-forms on $M$. Then $B^r(M, \mathbb{R}) \subset Z^r(M, \mathbb{R})$ which allows us to form the quotient space $H^r(M, \mathbb{R}) := Z^r(M, \mathbb{R})/B^r(M, \mathbb{R})$, called the **deRham cohomology group** of dimension $r$. Set

$$H^r(M, \mathbb{R}) = H^0(M, \mathbb{R}) \oplus H^1(M, \mathbb{R}) \oplus \cdots \oplus H^m(M, \mathbb{R}),$$

which is an algebra with the exterior multiplication.
Theorem (the deRham Theorem)  There is a natural isomorphism of $H^*(M, \mathbb{R})$ and the cohomology ring of $M$.

As an application of Hodge theory, we can study $H^r(M, \mathbb{R})$ using the nice representation of harmonic forms as follows

Theorem (Representing Cohomology Classes by Harmonic Forms). Each deRham cohomology class on $(M, g)$ contains a unique harmonic representative.

Proof. Let $h : \Lambda^r(M) \to \mathcal{H}^r(M)$ be the orthogonal projection. If $\omega \in \Lambda^r(M)$ is closed, then according to the Hodge decomposition, we have

$$\omega = d\alpha + h(\omega)$$

which implies that $[\omega] = [h(\omega)] \in H^r(M, \mathbb{R})$. Since $\mathcal{H}^r(M) \perp d\Lambda^{r-1}(M)$ we see that two different harmonic forms must belong to two different deRham cohomology classes. In fact, if $\gamma_1, \gamma_2 \in \mathcal{H}^r(M)$ and $[\gamma_1] = [\gamma_2]$, then $\gamma_1 - \gamma_2 = d\alpha$. But, $d\alpha \perp (\gamma_1 - \gamma_2)$, thus $d\alpha = 0$, so $\gamma_1 = \gamma_2$. Hence $h(\omega)$ is unique in $H^r(M, \mathbb{R})$.

From the proof of the Hodge theorem, we see that $\dim \mathcal{H}^r(M) < +\infty$ if $M$ is finite, so we get that $\dim H^r(M, \mathbb{R}) < +\infty$ if $M$ is compact.

Let $M$ be a compact, oriented, differentiable manifold of dimension $m$. We define a bilinear function

$$H^r(M, \mathbb{R}) \times H^{m-r}(M, \mathbb{R}) \to \mathbb{R}$$

by sending

$$([\phi], [\psi]) \mapsto \int_M \phi \wedge \psi.$$  

Observe that the bilinear map is well-defined, i.e. if $\phi_1 = \phi_1 + d\xi$, then, by Stoke’s theorem,

$$\int_M \phi_1 \wedge \psi = \int_M \phi \wedge \psi.$$
Theorem. Poincare duality theorem. The bilinear function above is non-singular pairing and consequently determines isomorphisms of $H^{m-r}(M)$ with the dual space of $H^r(M)$:

$$H^{m-r}(M, \mathbb{R}) \simeq (H^r(M, \mathbb{R}))^*.$$ 

In fact, given a non-zero cohomology class $[\phi] \in H^r(M, \mathbb{R})$, we must find a non-zero cohomology class $[\psi] \in H^{m-r}(M, \mathbb{R})$, such that $([\phi], [\psi]) \neq 0$. Choose a Riemannian structure. We can assume that $\phi$ is harmonic, and $\phi \neq 0$. Since $*\Delta = \Delta *$, we have that $*\phi$ is also harmonic, and $*\phi \in H^{m-r}(M, \mathbb{R})$. Now,

$$([\phi], [\psi]) = \int_M \phi \wedge *\phi = ||\phi||^2 \neq 0.$$ 

So the statement is proved.

The $r$-th Betti number $\beta_r(M)$ of $(M, g)$ is defined by

$$\beta_r(M) = \dim H^r(M, \mathbb{R}) = \dim \mathcal{H}^r.$$ 

Then we have

$$\beta_r(M) = \beta_{m-r}(M).$$ 

The Euler-Poincare characteristic number $\chi(M)$ of $(M, g)$ is defined by

$$\chi(M) = \sum_{r=0}^{m} (-1)^r \dim H^r(M, \mathbb{R}) = \sum_{r=0}^{m} (-1)^r \beta_r(M).$$ 

Then, we have the statement that if $m = \dim M$ is odd, then $\chi(M) = 0$.

Another statement we can prove (will be proved later) is Let $(M, g)$ be a compact oriented Riemannian manifold without boundary. If its Ricci curvature is positive, then

$$\beta_1(M) = \beta_{m-1}(M) = 0.$$