1 Differential forms, exterior operator and wedge and symmetric products

• Let $\omega_1, \omega_2$ be two 1-forms on $M$, then, for every two smooth tangent vector fields $X, Y$ on $M$,
  \[ \omega_1 \wedge \omega_2(X, Y) = \omega_1(X)\omega_2(Y) - \omega_1(Y)\omega_2(X). \]

• Let $\omega_1, \omega_2$ be two 1-forms on $M$, then the symmetric product of $\omega_1, \omega_2$, denoted by $\omega_1 \omega_2$ (denoted by juxtaposition with no product symbol), is
  \[ \omega_1 \omega_2 = \frac{1}{2}(\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1). \]

• Let $\omega$ be a one-form on $M$, then, for every two smooth tangent vector fields $X, Y$ on $M$,
  \[ d\omega(X, Y) = X(Y(\omega)) - Y(X(\omega)) - \omega([X, Y]). \]

  \textit{Proof:} Without loss of generality, we can just consider $\omega = f dg$. Hence $d\omega = df \wedge dg$. Then the identity is easily verified.

• Let $\omega_1$ be a $r$-form, then
  \[ d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^r \omega_1 \wedge d\omega_2. \]

2 Affine Connections for arbitrary frame fields

From Problem 4-5 and problem 7-2 on the textbook: Let $\nabla$ be a linear connection (a connection on $T(M)$). Let $\{E_i\}$ be a local frame on open subset $U \subset M$. Let $\{\phi^i\}$ be the dual co-frame. Write
  \[ \nabla_X E_i = \sum_{j=1}^{m} \omega_i^j(X)E_j. \]

$\omega_i^j$ are called \textit{connection forms}. The matrix $\omega = (\omega_i^j)$ is called the \textit{connection matrix} of $\nabla$ with respect to the local frame $\{E_i\}$. 
Define a matrix of 2-forms $\Omega = (\Omega^j_i)$ by
\[
\Omega^j_i = \frac{1}{2} \sum_{k,l=1}^{m} R^j_{ikl} \phi^k \wedge \phi^l,
\]
where
\[
R(E_k, E_l) E_i = R^j_{ikl} E_j.
\]

Structure equations:

The first structure equation
\[
d\phi^i = \sum_{j=1}^{m} \phi^j \wedge \omega^i_j + \tau^i;
\]
where $\{\tau^1, \ldots, \tau^n\}$ are the torsion 2-forms, defined by the torsion tensor $\tau$ and the frame $\{E_i\}$ by
\[
\tau(X, Y) = \tau^i(X, Y) E_i
\]
where
\[
\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].
\]
The first structure equation appeared in Problem 4-5 on the textbook).

The second structure equation
\[
d\omega^j_i = \sum_{h=1}^{m} \omega^h_i \wedge \omega^j_h + \frac{1}{2} \sum_{k,l=1}^{m} R^j_{ikl} \phi^k \wedge \phi^l
\]
where
\[
R(E_k, E_l) E_i = R^j_{ikl} E_j.
\]
The second structure equation appeared in Problem 7-2 on the textbook). The second structure equation can also be written as
\[
\Omega^j_i = d\omega^j_i - \sum_{h=1}^{m} \omega^h_i \wedge \omega^j_h,
\]
or simple
\[
\Omega = d\omega - \omega \wedge \omega.
\]
Proof: We first prove the first structure equation.

\[(d\phi^j - \sum_{j=1}^{m} \phi^j \wedge \omega^j)(E_k, E_l) = E_k(\phi^j(E_l)) - E_l(\phi^j(E_k)) - \phi^j([E_k, E_l])
- \sum_{j=1}^{m} [\phi^j(E_k)\omega^j(E_l) - \phi^j(E_l)\omega^j(E_k)]\]

\[= \omega^j_i(E_k) - \omega^j_i(E_l) - \omega^j([E_k, E_l])\]

\[= \Gamma^i_{lk} - \Gamma^i_{kl} - \phi^i([e_k, e_l])\]

\[= \phi^i(\nabla E_k E_l - \nabla E_l E_k - [E_k, E_l])\]

Hence, the first identity has been proven.

We now prove the second identity.

\[(d\omega^j_i - \sum_{h=1}^{m} \omega^h_i \wedge \omega^h_j)(E_k, E_l) = E_k(\omega^j_i(E_l)) - E_l(\omega^j_i(E_k))
- \omega^j_i([E_k, E_l]) - \sum_{h=1}^{m} [\omega^h_i(E_k)\omega^h_j(E_l) - \omega^h_i(E_l)\omega^h_j(E_k)]\]

\[= E_k(\Gamma^j_{il}) - E_l(\Gamma^j_{ik}) - \sum_{h=1}^{m} \omega^h_i([E_k, E_l])\Gamma^j_{ih}\]

\[- \sum_{h=1}^{m} [\Gamma^h_i \Gamma^j_{il} - \Gamma^h_l \Gamma^j_{ih}]\].

On the other hand,

\[R(E_k, E_l)E_i = \nabla E_k \nabla E_l E_i - \nabla E_l \nabla E_k E_i - \nabla [E_k, E_l]E_i\]

\[= \sum_{j=1}^{m} [\nabla E_k (\Gamma^j_{il} E_j) - \nabla E_l (\Gamma^j_{ik} E_j)] - \sum_{h=1}^{m} \phi^h([E_k, E_l])\Gamma^j_{ih} E_j\]

\[= \sum_{j=1}^{m} [\nabla E_k (\Gamma^j_{il}) - \nabla E_l (\Gamma^j_{ik}) + \sum_{h=1}^{m} (\Gamma^h_i \Gamma^j_{hk} - \Gamma^h_k \Gamma^j_{hi} - \phi^h([E_k, E_l])\Gamma^j_{ih})] E_j\]

Hence we have

\[R(E_k, E_l)E_i = \sum_{j=1}^{m} [(d\omega^j_i - \sum_{h=1}^{m} \omega^h_i \wedge \omega^h_j)(E_k, E_l)] E_j.\]
Thus

\[ R^j_{ikl} = \omega^j(R(E_k, E_l)e_i) = (d\omega^j - \sum_{h=1}^{m} \omega^j_h \wedge \omega^i_h)(E_k, E_l). \]

The second identity is thus proved.

- **Bianchi indentity:**

  \[
  d\Omega = \omega \wedge \Omega - \Omega \wedge \omega,
  \]

  \[
  d\Omega^j = \sum_{k=1}^{m} \omega^k \wedge \Omega^j_k - \sum_{h=1}^{m} \Omega^h_k \wedge \omega^j_h,
  \]

  or simply we can write

  \[
  d\Omega = \omega \wedge \Omega - \Omega \wedge \omega.
  \]

  The proof of Bianchi indentity is as follows: from the second structure equation,

  \[ \Omega = d\omega - \omega \wedge \omega, \]

  Hence

  \[
  d\Omega = -d\omega \wedge \omega + \omega \wedge d\omega
  \]

  \[ = -(\Omega + \omega \wedge \omega) \wedge \omega + \omega(\Omega + \omega \wedge \omega) = -\Omega \wedge \omega + \omega \wedge \Omega. \]

- **Cartan's theory** Let \( M \) be a Riemannian manifold with Riemannian metric \( ds^2 \). Let \( \{\omega_i, 1 \leq i \leq m\} \) be a set of 1-forms on \( U \) which are linearly independent at every point in \( U \), and such that

  \[
  ds^2 = \omega_1^2 + \cdots + \omega_m^2.
  \]

Then there exists a unique set of \( m^2 \) 1-forms \( \{\omega_{ij}\} \) with \( \omega_{ij} = -\omega_{ji} \), called the connection forms, such that

\[
 d\omega_i = \sum_{j=1}^{m} \omega_j \wedge \omega_{ij};
\]

\[
 d\omega_{ij} = \sum_{k=1}^{m} \omega_{kj} \wedge \omega_{ik} + \frac{1}{2} R_{ijkl} \omega_k \wedge \omega_l.
\]
The tensor $R_{ijkl}$ is called the curvature tensor of the Riemannian metric $ds^2$.

Note that these structure equations are often used together with the Bianchi identity:

$$d\Omega = \omega \wedge \Omega - \Omega \wedge \omega.$$

### 3 Calculating the sectional curvature

Recall that

$$K_p(E) = -\frac{R(X,Y,X,Y)}{G(X,Y,X,Y)}.$$

Let $e_1, \ldots, e_m$ be an orthonormal (with respect to the Riemannian metric) local frame field for $T(M)$ on $U$ with $e_1, e_2 \in E$. Let $\omega^1, \ldots, \omega^m$ be its dual (they are one-forms). Then $G = \sum_{i=1}^m \omega^i \otimes \omega^i$. So $G(e_1, e_2, e_1, e_2) = 1$. Thus

$$K_p(E) = -R(e_1, e_2, e_1, e_2).$$

where $R(e_1, e_2, e_1, e_2) = G(\mathcal{R}(e_1, e_2)e_1, e_2)$. Write

$$R(e_1, e_2)e_1 = \sum_{j=1}^m \omega^j(\mathcal{R}(e_1, e_2)e_1)e_j.$$

Then $G(\mathcal{R}(e_1, e_2)e_1, e_2) = \omega^2(\mathcal{R}(e_1, e_2)e_1)$. Hence

$$K_p(E) = -\omega^2(\mathcal{R}(e_1, e_2)e_1).$$

Recall the (second) structure equation (taking $i = 1, j = 2$):

$$d\omega^2_1 = \sum_{h=1}^m \omega^2_h \wedge \omega^2_h + \frac{1}{2} \sum_{k,l=1}^m \omega^2(\mathcal{R}(e_k, e_l)e_1)\omega^k \wedge \omega^l.$$

This means the sectional curvature $K_p(E)$ can calculated by calculating the connection forms $\omega^k_j$, calculating $d\omega^2_1$, and using the (second) structure equation.

**Example:** On $\mathbb{R}^m$, define

$$G = \frac{4}{(1 + c \sum_{i=1}^m (x^i)^2)^2} \sum_{i=1}^m dx^i \otimes dx^i.$$
where $c > 0$ is a constant. Calculate its sectional curvature.

**Solution:** Let $A = 1 + c \sum_{i=1}^{m} (x^i)^2$, then $g_{ij} = G(\partial/\partial x^i, \partial/\partial x^j) = 0$ if $i \neq j$ and $g_{ii} = \frac{2}{A}$. So let $e_i = (A/2) \partial/\partial x^i$. Then $\{e_1, \ldots, e_m\}$ are the orthonormal basis (with respect to this Riemannina metric). Let $\{\omega^1, \ldots, \omega^m\}$ be its dual basis. So have $\omega^i = \frac{2}{A} dx^i$. We can write

$$G = \sum_{i=1}^{m} \omega^i \otimes \omega^i.$$ 

Let $E \subset T_p \mathbb{R}^m$ be a subspace of dimension two, without loss of generality, we assume that $e_1, e_2 \in E$. From above, to calculate the sectional curvature, we only need to calculate $\omega^2(\mathcal{R}(e_1, e_2)e_1)$. To do this, we use the structure equations,

$$d\omega^2 = \sum_{h=1}^{m} \omega^h \wedge \omega^2_h + \frac{1}{2} \sum_{k,l=1}^{m} \omega^2(\mathcal{R}(e_k, e_l)e_1)\omega^k \wedge \omega^l.$$ 

We first find the connection forms $\omega^i_j$ (using the fundamental theorem of Riemannian geometry). In fact,

$$d\omega^i = 2\frac{dx^i}{A^2} \wedge dA = \frac{4c}{A^2} \sum_j x^j dx^i \wedge dx^j = \sum_j x^j \omega^i \wedge \omega^j.$$ 

On the other hand, by structure equation

$$d\omega^i = \sum_j \omega^j \wedge \omega^i_j, \quad \omega^i_j + \omega^j_i = 0.$$ 

To find $\omega^i_j$, write (since $\omega^i, 1 \leq i \leq m$, is a basis)

$$\omega^i_j = \sum_{k=1}^{m} A^{i,j}_k \omega^k.$$ 

From $\omega^i_j + \omega^j_i = 0$, We know that $\omega^i_i = 0$ and we know that $A^{i,j}_k = -A^{j,i}_k$, $a^{i,i}_k = 0$. Hence

$$d\omega^i = \sum_j \omega^j \wedge \omega^i_j = \sum_j \omega^j \wedge (\sum_{k=1}^{m} A^{i,j}_k \omega^k) = \sum_{k,j=1}^{m} A^{i,j}_k \omega^j \wedge \omega^k = \sum_{j<k} (A^{i,j}_k - A^{i,k}_j) \omega^j \wedge \omega^k.$$ 


From above \( d\omega^i = c \sum_j x^j \omega^i \wedge \omega^j \). Hence
\[
c \sum_j x^j \omega^i \wedge \omega^j = \sum_{j<k} (A^{ij}_k - A^{ik}_j) \omega^j \wedge \omega^k.
\]
This means that, by looking at the terms before \( x^i \wedge x^k \) for \( k > i \), \( A^{ij}_k - A^{ik}_j = cx^k \), since \( A^{ij}_k = 0 \), we have \( A^{ij}_k = -cx^k \). By looking at the terms before \( x^i \wedge x^j \) for \( j < i \), we have \( A^{ij}_i - A^{ij}_k = cx^j \). Hence \( A^{ij}_k = cx^j \) for \( j < i \).
For any \( k, j \neq i \), by comparing both sides of the equation, we have \( A^{ij}_k = 0 \).
Hence we get the connection forms
\[
\omega^j = c(x^i \omega^j - x^j \omega^i).
\]
In particular,
\[
\omega^2 = c(x^2 \omega^1 - x^1 \omega^2).
\]
To calculate the curvature, we use the second structure equation. To do so, we need to calculate \( d\omega^2 - \sum_{h=1}^m \omega^h \wedge \omega^2_h \). In fact,
\[
d\omega^2 - \sum_{h=1}^m \omega^h \wedge \omega^2_h = c(dx^2 \wedge \omega^1 + x^2 dx^1 \wedge \omega^2 - x^i dx^2) \]
\[
= c \left( A\omega^2 \wedge \omega^1 + c \sum_{k=1}^m x^k x^j \omega^1 \wedge \omega^k \right) - c^2 \sum_{h=1}^m \left( x^h x^j \omega^1 \wedge \omega^h + x^j x^h \omega^2 \wedge \omega^1 \right) \]
\[
= (Ac - c^2 \sum_{k=1}^m (x^k)^2) \omega^2 \wedge \omega^1 = -c\omega^1 \wedge \omega^2.
\]
From the structure equation, we have
\[
d\omega^2 - \sum_{h=1}^m \omega^h \wedge \omega^2_h = \frac{1}{2} \sum_{k,l=1}^m \omega^2(R(e_k, e_l)e_1) \omega^k \wedge \omega^l.
\]
Hence
\[
-c\omega^1 \wedge \omega^2 = \frac{1}{2} \sum_{k,l=1}^m \omega^2(R(e_k, e_l)e_1) \omega^k \wedge \omega^l.
\]
Which means that $\omega^2(R(e_1, e_2)e_1) = -c$. we have

$$K_p(E) = -\omega^2(R(e_1, e_2)e_1) = c.$$ 

Hence its sectional curvature is constant.

**Remark:** An alternative way to find the connection forms $\omega^k_j$ is to use the uniqueness of existence of Riemannina connection, as follows:

$$d\omega^i = 2\frac{dx^i}{A^2} \wedge dA = \frac{4c}{A^2} \sum_j x^j dx^i \wedge dx^j = \frac{c}{A^2} \sum_j x^j \omega^i \wedge \omega^j = \sum_j \omega^i \wedge c(x^j \omega^i - x^i \omega^j).$$

If we let $\omega^i_j = c((x^i \omega^j - x^j \omega^i)$, then

$$d\omega^i = \omega^j \wedge \omega^i_j, \quad \omega^i_j + \omega^j_i = 0.$$

By the uniqueness of the existence of Riemannina connection, we have $(\omega^i_j)$ is the connection matrix of the L-C connection $\nabla$ with respect to the frame $\omega^i$. Hence $\omega^i_j = c((x^i \omega^j - x^j \omega^i)$ are the connection forms.

**HW:** Let $U \subset \mathbb{R}^{n-1}$ be an open set. Let $\eta_2, \ldots, \eta_n$ be smooth one forms on $U$ such that

$$(ds^1)^2 = \eta_2^2 + \cdots + \eta_n^2$$

defines a Riemannian metric on $U$. Let $\theta_2, \ldots, \theta_n$ be local coordinates of $U$. Let $f(r, \theta)$ be a positive function on $\mathbb{R}^+ \times U$. Define a Riemannian metric

$$ds^2 = dr^2 + f^2(r, \theta) \sum_{i=2}^{n} \eta_i^2$$
on $\mathbb{R}^+ \times U$, where $\mathbb{R}^+$ is the set of positive real numbers. Remark: example of this setting appears in the Euclidean metric under polar coordinates. For example, in $\mathbb{R}^2$, the Euclidean metric can be written as $ds^2 = dr^2 + r^2 d\theta^2$ (of course the curvature in this case is zero). Calculate the curvature tensors as follows:

**Step 1:** Let $\omega_1 = dr, \omega_k = f(r, \theta)\eta_k, k > 1$. Then

$$ds^2 = \omega_1^2 + \cdots \omega_n^2.$$
Let $e_1, \ldots, e_n$ be the dual basis of $\omega_1, \ldots, \omega_n$, and let $\eta_{kl}$ be the connection forms of $(ds^1)^2$ (which is the metric on $U$). Define 

$$\omega_{1l} = -e_1(\log f) \omega_l$$

and 

$$\omega_{kl} = \eta_{kl} - e_k(\log f) \omega_l + e_l(\log f) \omega_k$$

for $k, l > 1$. Do the following proofs:

1. Prove that $d\omega_i = -\omega_{ij} \wedge \omega_j$
2. Show that 

$$d\omega_{1l} + \omega_{1k} \wedge \omega_{kl} = -\frac{e_k(\partial f/\partial r)}{f} \omega_k \wedge \omega_l,$$

using the fact that $e_1 = \partial/\partial r$.

Use (2) to show that 

$$R_{1lmn} = -\delta_{ln} \frac{e_m(\partial f/\partial r)}{f} + -\delta_{lm} \frac{e_n(\partial f/\partial r)}{f}.$$

In particular, the Ricci curvature at $\partial/\partial r$ direction is

$$R_{11} = R_{11l} = -(n - 1) \frac{1}{f} \frac{\partial^2 f}{(\partial r)^2}.$$