Riemannian Geometry
The Bochner-Weitzenbock formula

If we need to verify some tensor identity (or inequality) on Riemannian manifolds, we only need to choose, at every point, a suitable local coordinate, and verify the identity (or inequality) at the given point. In this handout, we will discuss how to make the choice of the local coordinate and prove (or re-prove) some useful formulas for the differential operators on the Riemannian manifolds $M$.

1 Normal (geodesic) coordinates

We first prove the following statement about the existence of the normal coordinates: For each point $p \in M$, there exists the normal coordinate $(U, x^1, \ldots, x^m)$ at $p$, i.e. we have $x^i(p) = 0$, $g_{ij}(p) = \delta_{ij}$ and $\Gamma^k_{ij}(p) = 0$ for $1 \leq i, j, k \leq m$.

**Proof:** According the theorem about existence of the solution for (system) ODEs, for every point $P \in M$ and $v \in T_P(M)$, there exists a unique geodesics $C(t, P, v)$ (or we just write $C_v(t)$) such that $C_v(0) = P$, $C'_v(0) = v$. Before we continue, we make the following very important remark: we always have $C_{\lambda v}(t) = C_v(\lambda t)$, because if we set $\alpha(u) = C_v(\lambda u)$, then

$$\frac{d\alpha}{du} = \lambda \frac{dC_v}{dt}, \quad \frac{d^2\alpha}{du^2} = \lambda^2 \frac{d^2C_v}{dt^2}.$$  

Thus, since $C_v$ is a geodesic, $\alpha$ is also a geodesic and satisfies that $\alpha(0) = P, \alpha'(0) = \lambda v$, so by the uniqueness, $C_v(\lambda t) = C_{\lambda v}(t)$. This proves the remark.

The exponential mapping $\exp_P(v) : T_P(M) \to M$ is defined by $v \mapsto C_v(1)$. Then, we can prove that there exists $\epsilon > 0$ such that $\exp_P : B(O_P, \epsilon) \subset T_P(M) \to M$ is a diffeomorphism onto its image. Let $U = \exp_P(B(O_P, \epsilon))$. Take an orthonormal basis $\{e_1, \ldots, e_m\}$ for $T_P(M)$, this determines an associated coordinate $(x^1, \ldots, x^m)$. We verify that the coordinates $(U, x^1, \ldots, x^m)$ is a normal coordinate for $P$. First, we claim that

$$\frac{\partial}{\partial x^i}|_P = e_i.$$
In fact, let $\phi = (x^1, \ldots, x^m)$, and take any function $f$ around $P$, we have (only verify for $i = 1$)

$$\frac{\partial f}{\partial x^1}|_P = \frac{\partial f \circ \phi^{-1}}{\partial x^1}|_P$$

$$= \frac{d}{dt} f \circ \phi^{-1}(t, 0, \ldots, 0)|_{t=0} = \frac{d}{dt} f \circ C_{e_1}(1)|_{t=0} = \frac{d}{dt} f \circ C_{e_1}(t)|_{t=0} = e_1(f).$$

So

$$\frac{\partial}{\partial x^i}|_P = e_i.$$

This implies that $g_{ij}(p) = \delta_{ij}$.

Next, we show that $\Gamma^k_{ij}(p) = 0$ for $1 \leq i, j, k \leq m$. Let $q = C_\xi(1) \in U$ and let $C^i(t)$ be the coordinate components of $C_\xi(t)$. Write $\phi(q) = (\xi^1, \ldots, \xi^m)$. Then, since $C_\xi(t) = C_\xi(1)$, we have that $C^i(t) = t\xi^i$. Since $C_\xi(t)$ is geodesic, the geodesic equation implies that

$$\Gamma^k_{ij}(C_\xi(t))\xi^i\xi^j = 0.$$

Let $t \to 0$, we get

$$\Gamma^k_{ij}(C_\xi(0))\xi^i\xi^j = 0.$$

Because $\xi^i$ are arbitrary, and $\Gamma^k_{ij} = \Gamma^k_{ji}$, this implies that $\Gamma^k_{ij}(p) = 0$. So the statement is proved.

2 Reproof of some formulas for Differential operators

In this notes, we always assume that $M$ is a Riemannian manifold with the Levi-Civita connection $\nabla$, and $X, Y, \ldots$ are smooth vector fields.

- A basis formula for $d$. Let $(M, g)$ be a Riemannian manifold with the Levi-Civita connection $\nabla$. Let $\{e_i\}$ be a local frame field on $M$ (i.e. a basis for $\Gamma(U, TM)$) and $\{\omega^i\}$ be its dual, i.e. $\omega^i(e_j) = \delta^i_j$. Then, for every smooth differential $r$-form $\theta$,

$$d\theta = \sum_i \omega^i \wedge \nabla_{e_i} \theta. \quad (1)$$
Proof: First notice that it is independent of the choice of coordinates. So we choose normal coordinates $x^i$, i.e. we have $x^i(p) = 0$, $g_{ij}(p) = \delta_{ij}$ and $\Gamma^k_{ij}(p) = 0$ for $1 \leq i, j, k \leq m$. Let $\{e_i\}$ is a local orthonormal frame field on $M$ with $e_i(p) = \frac{\partial}{\partial x^i}|_p$, and let $\{\omega^j\}$ be the dual to $\{e_i\}$. We claim that
\[(\nabla_{e_i} \omega^j)(p) = 0.\]
In fact, since $\delta_{jk} = \omega^j(e_k) = (e_k, \omega^j)$, we have
\[d\delta_{jk} = 0 = (\nabla e_k, \omega^j) + (e_k, \nabla \omega^j),\]
i.e. $\nabla \omega^j(e_k) = \omega^j(\nabla e_k)$, hence
\[\nabla_{e_i} \omega^j = -\sum_{k=1}^{m} \Gamma^j_{ik} \omega^k.\]
Thus we get, using $\Gamma^j_{ik}(p) = 0$,
\[\nabla_{e_i} \omega^j(p) = 0.\]
Therefore the claim holds.

Now, since $\nabla_{\partial/\partial x^i} dx^j = 0$, we have, for $\theta = f dx^{i_1} \wedge \cdots \wedge dx^{i_r}$,
\[\sum_i \omega^j \wedge \nabla_{e_i} \theta = \sum_i \frac{\partial f}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r} = d\theta.\]

**The divergence operator.** A $(1,1)$-tensor $T$ can be viewed as an endomorphism $T : V \rightarrow V$, and the trace of $T$ is called the contraction of $T$, denoted by $\text{trace}(T)$ or $C^1_1(T)$. Let $X \in \Gamma(TM)$. Then $\nabla X$ is a smooth $(1,1)$-tensor field on $M$. Take a contraction of $\nabla X$, we get a smooth function on $M$. It is called the divergence of $X$, i.e. $\text{div}(X) = C^1_1(\nabla X)$. The map $\text{div} : \Gamma(TM) \rightarrow C^\infty(M)$ given by $X \mapsto \text{div}(X)$ is called the divergence operator. In terms of local coordinate $(U; x^i)$,
\[\text{div}(X) = \frac{1}{\sqrt{G}} \sum_{i=1}^{m} \frac{\partial}{\partial x^i}(\sqrt{G} X^i),\]
where $G = \det(g_{ij})$, $g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j)$, $X^i = dx^i(X)$. 

3
Solution: Choose normal coordinates \( x^i \) at \( p \in M \) and write
\[
X|_U = X^i \frac{\partial}{\partial x^i}.
\]
Then, at point \( p \),
\[
\frac{1}{\sqrt{G}} \sum_{i=1}^m \frac{\partial}{\partial x^i}(\sqrt{G} X^i) = \sum_{i=1}^m \frac{\partial X^i}{\partial x^i},
\]
where on the other hand, at point \( p \),
\[
div(X) = tr\{Y \rightarrow \nabla_Y X\} = \sum_{i=1}^m \nabla_{\frac{\partial}{\partial x^i}} X^i = \sum_{i=1}^m \frac{\partial X^i}{\partial x^i}.
\]
So the formula holds.

- **The gradient of** \( f \) Let \( f \in C^\infty(M) \), define a tangent vector field \( \text{grad}(f) \) on \( M \), by
\[
g(\text{grad}(f), X) = df(X) = X(f),
\]
for every smooth tangent vector field \( X \). The tangent vector field \( \text{grad}(f) \) is called the **gradient** of \( f \). In local coordinate \((U; x^i)\),
\[
\text{grad}(f) = \sum_{j=1}^m \left( \sum_{i=1}^m g^{ij} \frac{\partial f}{\partial x^i} \right) \frac{\partial}{\partial x^j},
\]
where \( G = \det(g_{ij}), g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j), (g^{ij}) = (g_{ij})^{-1}. \)

- **Beltrami-Laplace operator** Let \( f \in C^\infty(M) \), define \( \Delta f = \text{div}(\text{grad}(f)) \). It is called the **Laplace operator**. In local coordinate \((U; x^i)\),
\[
\Delta f = \frac{1}{\sqrt{G}} \sum_{i=1}^m \frac{\partial}{\partial x^i} \left( \sum_{j=1}^m \sqrt{G} g^{ij} \frac{\partial f}{\partial x^j} \right),
\]
where \( G = \det(g_{ij}), g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j), (g^{ij}) = (g_{ij})^{-1}. \)

- **Interior product** For any vector field \( X, \iota(X) \) sends \( r \)-form to \( r-1 \) defined by, for every \( r \)-form \( \omega \) and vector fields \( Y_1, \ldots, Y_{r-1} \),
\[
(\iota(X)\omega)(Y_1, \ldots, Y_{r-1}) = \omega(X, Y_1, \ldots, Y_{r-1}).
\]
• Let \( \eta \) be its volume form of \( M \). For every smooth tangent vector field \( X \),
\[
d(\iota(X)\eta) = \text{div}(X)\eta,
\]
where \( \iota(X) \) is the interior product.

**Proof:** By definition, \( \eta = \sqrt{G}dx^1 \wedge \cdots \wedge dx^m \), we claim \( \iota(X)\eta = \omega \) where
\[
\omega = \sum_{i=1}^{m} (-1)^{i+1} \sqrt{g}X^i dx^1 \wedge \cdots \wedge \hat{dx}^i \wedge \cdots \wedge dx^m.
\]
We now prove it. Indep. of the choice of coordinates. Choose normal coordinates \( x^i \). Then, at the point \( p \), \( \eta = dx^1 \wedge \cdots \wedge dx^m \) and \( X = \sum_{j=1}^{m} X^j e_j \), hence
\[
\iota(e_j)(\omega^1 \wedge \cdots \wedge \omega^m) = (-1)^{j+1} \omega^1 \wedge \cdots \hat{\omega}^j \wedge \cdots \wedge \omega^m.
\]
This proves the claim. The rest of proof follows easily.

• **Divergence theorem:** Let \((M, g)\) be a compact oriented Riemannian manifold, then, for every smooth tangent vector field \( X \),
\[
\int (\text{div}X)\eta = 0,
\]
where \( \eta \) is the volume form.

• **The expression of the co-differential operator \( \delta \):** Let \( \{e_i\} \) be a local frame field on \( M \) compatible with the orientation of \( M \). Let \( g_{ij} = g(e_i, e_j) \), and \( (g^{ij}) = (g_{ij})^{-1} \). Then the codifferential operator \( \delta \) can be written as
\[
\delta \alpha = - \sum_{i,j=1}^{m} g^{ij}(\nabla_{e_i} \alpha), \quad \text{for every } \alpha \in \Lambda^r(M).
\]

If \( \{e_i\} \) is **orthonormal**, then we can write
\[
\delta = - \sum_{j=1}^{m} \iota(e_j)\nabla_{e_j}, \quad (2)
\]
where \( i(X) \) is the interior product operator, i.e. for every \( \alpha \in \Lambda^r(M) \), and for every tangent vector fields \( X_1, \ldots, X_{r-1} \).

\[
\delta \alpha(X_1, \ldots, X_{r-1}) = - \sum_{j=1}^{m} \iota(e_j)(\nabla e_j \alpha)(X_1, \ldots, X_{r-1}) \\
= - \sum_{j=1}^{m} (\nabla e_j \alpha)(e_j, X_1, \ldots, X_{r-1}).
\]

**Proof:** For \( p \in M \), choose the normal coordinate \((U, x^1, \ldots, x^m)\) at \( p \). Let \( \{e_i\} \) is a local orthonormal frame field on \( M \) with \( e_i(p) = \frac{\partial}{\partial x^i}|_p \), and let \( \{\omega^j\} \) be the dual to \( \{e_i\} \). Then

\[
(\nabla e_j \omega^j)(p) = 0.
\]

To prove

\[
\delta(\alpha) = - \sum_{j=1}^{m} \iota(e_j)(\nabla e_j \alpha). \tag{3}
\]

We need only to verify it at each point \( p \in M \). Since the operator is linear, without loss of generality, we assume that

\[
\alpha = f \omega^1 \wedge \cdots \wedge \omega^r.
\]

Hence

\[
\nabla e_j \alpha = \nabla e_j (f) \omega^1 \wedge \cdots \wedge \omega^r + f \nabla e_j (\omega^1 \wedge \cdots \wedge \omega^r)
= e_j (f) \omega^1 \wedge \cdots \wedge \omega^r + f \nabla e_j (\omega^1 \wedge \cdots \wedge \omega^r).
\]

Using \((\nabla e_j \omega^j)(p) = 0, (\text{only})\) at the point \( p \), we have

\[
\nabla e_j \alpha = e_j (f) \omega^1 \wedge \cdots \wedge \omega^r.
\]

Hence, at the point \( p \), we have

\[
\iota(e_j)(\nabla e_j \alpha) = e_j (f) \iota(e_j)(\omega^1 \wedge \cdots \wedge \omega^r).
\]

Because

\[
\iota(e_j)(\omega^1 \wedge \cdots \wedge \omega^r) = (-1)^{j+1} \omega^1 \wedge \cdots \wedge \omega^j \wedge \cdots \wedge \omega^r,
\]

6
we have
\[ \iota(e_j)(\nabla e_j \alpha) = (-1)^{j+1} e_j(f) \omega^1 \wedge \cdots \wedge \hat{\omega}^j \wedge \cdots \wedge \omega^r. \]

This tells us, at the point \( p \), that
\[
- \sum_{j=1}^m \iota(e_j)(\nabla e_j \alpha) = - \sum_j (-1)^{j+1} e_j(f) \omega^1 \wedge \cdots \wedge \hat{\omega}^j \wedge \cdots \wedge \omega^r. \tag{4}
\]

We now calculate the left-hand side. By definition, \( \delta = (-1)^{n(r+1)+1} \ast d \ast \).
We have,
\[
\delta(\alpha) = (-1)^{n(r+1)+1} \ast d \ast (\alpha) = (-1)^{n(r+1)+1} \ast d \ast (f \omega^1 \wedge \cdots \wedge \omega^r) \\
= (-1)^{n(r+1)+1} \ast d(f \omega^{r+1} \wedge \cdots \wedge \omega^m) \\
= (-1)^{n(r+1)+1} \ast \sum_j e_j(f) \omega^j \wedge \omega^{r+1} \wedge \cdots \wedge \omega^m.
\]

Note that
\[
\ast(\omega^j \wedge \omega^{r+1} \wedge \cdots \wedge \omega^m) = (-1)^{(r-1)(n-r-1)+(r-j)} \omega^1 \wedge \cdots \wedge \hat{\omega}^j \wedge \cdots \wedge \omega^r,
\]
hence
\[
\delta(\alpha) = \sum_j (-1)^{n(r+1)+1}(-1)^{(r-1)(n-r-1)+(r-j)} e_j(f) \omega^1 \wedge \cdots \wedge \hat{\omega}^j \wedge \cdots \wedge \omega^r \\
= \sum_j (-1)^j e_j(f) \omega^1 \wedge \cdots \wedge \hat{\omega}^j \wedge \cdots \wedge \omega^r.
\]

Comparing the above identity with (4), we conclude that (2) holds at every point \( p \). Hence the theorem holds.

- The operator \( \delta \) can be viewed as a generalization of the divergence: In fact, let \( X \) be a vector field, then \( \delta(\alpha_X) = -div(X) \), where \( \alpha_X \) is the 1-form defined by \( \alpha_X(Y) = g(X, Y) \) for every smooth tangent vector field \( Y \).

\textit{Proof:} Again, they are independent of the choice of coordinates. Choose normal coordinates \( x^i \). Then, at point \( p \). Let \( \{e_i\} \) is a local orthonormal frame field on \( M \) with \( e_i(p) = \frac{\partial}{\partial x^i} \big|_p \), and let \( \{\omega^j\} \) be the dual to \( \{e_i\} \).
Let $X = \sum X^i e_j$, then $\alpha_X = \sum X^j \omega^j$, $div(X) = -\sum e_i(X^i)$. On the other hand,
\[ \delta(\alpha_X) = -\sum \nabla_{e_i}(\alpha_X)(e_i). \]
At point $p$,
\[ \nabla_{e_i}(\alpha_X) = \sum e_i(X^j) \omega^j + \sum X^j \nabla_{e_i} \omega^j = \sum e_i(X^j) \omega^j. \]
Hence $\nabla_{e_i}(\alpha_X)(e_i) = e_i(X^i)$. This proves the theorem.

3 Bochner-Weitzenbock Formulas

The Bochner-Weitzenbock Formulas, sometimes referred to as the Bochner technique, is one of the most important techniques in the theory of geometric analysis.

We want to express the Hodge-Laplace operator $\triangle$ in terms of the Levi-Civita connection $\nabla$.

We first consider the function case. For functions, i.e. 0-form $f$, by definition, $g(\text{grad}(f), Y) = Y(f)$ and $df(Y) = Y(f)$, so $\alpha_{\text{grad}(f)} = df$. Hence, from above, $\delta(df) = -\text{div}(\text{grad}(f))$. Therefore,
\[ \triangle f = \delta df = -\text{div}(\text{grad}(f)) = -tr \nabla^2 f, \]
where, $tr \nabla^2 := \sum_{i=1}^m \nabla_{e_i} \nabla_{e_i}$ for an orthonormal basis $\{e_i\}$.

In general, let $\{e_i\}$ be a local frame for a Riemannian manifold $(M, g)$, define
\[ tr \nabla^2 : \Lambda^r(M) \rightarrow \Lambda^r(M) \]
as
\[ tr \nabla^2 (\alpha) = g^{ij} (\nabla_{e_i} \nabla_{e_j} - \nabla_{\nabla_{e_i} e_j}) \alpha, \]
for every $\alpha \in \Lambda^r(M)$.

For 1-form $\alpha$, we have
\[ \triangle \alpha(X) = -tr \nabla^2 \alpha(X) + r(\alpha^\#, X), \]
where $r$ is the Ricci tensor of $(M, g)$, and $\alpha^\#$ is the vector field defined by $g(\alpha^\#, Y) = \alpha(Y)$.

**Proof.** The right hand side is independent of the choice of our orthonormal frame field. Therefore, we only need to verify it at every point $p \in M$. To do so, we choose normal coordinates centered at $p$ and put at $p$,

$$e_i = \frac{\partial}{\partial x^i}.$$ 

Let $\omega^j$ be its dual frame. Then, always at $p$,

$$\nabla_{e_i} e_j = 0.$$ 

This also gives, at $p$,

$$\nabla_{e_i} \omega^j = 0.$$ 

Using (1) and (2), we then have at $p$,

$$(\delta d\alpha)(X) = (\delta(\omega^j \wedge \nabla_{e_j} \alpha))(X) = -(\nabla_{e_i}(\omega^j \wedge \nabla_{e_j} \alpha))(e_i, X) = -(\omega^j \wedge \nabla_{e_i} \nabla_{e_j} \alpha)(e_i, X).$$

Using (1) and (2), we then have at $p$,

$$(d\delta\alpha)(X) = (\omega^j \nabla_{e_j} (\delta\alpha))(X) = X^k \nabla_{e_j} (\delta\alpha) = -X^k \nabla_{e_j} \nabla_{e_i} \alpha.$$ 

Hence,

$$\Delta \alpha(X) = -tr \nabla^2 \alpha(X) + X^j \nabla_{e_i} \nabla_{e_j} \alpha(e_i) - X^j \nabla_{e_j} \nabla_{e_i} \alpha) = X^j R(e_i, e_j) \alpha(e_i),$$

where where $X = X^k e_k$ and

$$R(e_i, e_j) \alpha = (\nabla_{e_i} \nabla_{e_j} - \nabla_{e_j} \nabla_{e_i}) \alpha.$$ 

We now claim that

$$X^j R(e_i, e_j) \alpha(e_i) = r(\alpha^\#, X).$$
In fact, write $\alpha = \alpha_k \omega^k$, then $\alpha^\# = \alpha_k e_k$, and

\[
X^j R(e_i, e_j) \alpha(e_i) = X^j \alpha_k R(e_i, e_j) \omega^k(e_i)
\]
\[
= -X^j \alpha_k R(e_j, e_i) \omega^k(e_i)
\]
\[
= -X^j \alpha_k R_{kmji} \omega^m(e_i)
\]
\[
= -X^j \alpha_k R_{bijji}
\]
\[
= -r(\alpha^\#, X).
\]

This proves the statement.

We now derive the formula for a general $r$-form. For the purpose, we define the second covariant derivative as

\[
\nabla_2^2 = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}.
\]

**Weitzenbock’s formula.** Let $e_1, \ldots, e_m$ be a local orthonormal frame field, with the dual frame field $\omega^1, \ldots, \omega^m$. Then the Hodge-Laplace operator $\Delta$ acting on $r$-differential forms is given by

\[
\Delta = -\sum_{i=1}^m \nabla_{e_i e_i}^2 - \sum_{i,j} \omega^i \wedge \iota(e_j) R(e_i, e_j).
\]

**Proof.** The right hand side is independent of the choice of our orthonormal frame field. Therefore, we only need to verify it at every point $p \in M$. To do so, we choose normal coordinates centered at $p$ and put at $p$,

\[
e_i = \frac{\partial}{\partial x^i}.
\]

Then, always at $p$,

\[
\nabla_{e_i} e_j = 0.
\]

Hence,

\[
\nabla_{e_i e_i}^2 = \nabla_{e_i} \nabla_{e_i} ;
\]

and also

\[
\left( \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^j} \right) = 0.
\]
Therefore

\[ R(e_i, e_j) = \nabla_{e_i} \nabla_{e_j} - \nabla_{e_j} \nabla_{e_i}. \]

Using (1) and (2), we then have at \( p \),

\[ \delta d = -\iota(e_j) \nabla_{e_j} (\omega^i \wedge \nabla_{e_i}) \]

\[ = -\iota(e_j)(\omega^i \wedge \nabla_{e_j} \nabla_{e_i}) \]

\[ = -\nabla_{e_j} \nabla_{e_j} + \omega^i \wedge \iota(e_j) \nabla_{e_j} \nabla_{e_i}. \]

To calculate \( d\delta \), we note that, since \( \nabla_{e_i} \omega^j = 0 \),

\[ \iota(e_j) \nabla_{e_i} = \nabla_{e_i} \iota(e_j). \]

Hence,

\[ d\delta = -\omega^i \wedge \nabla_{e_i} (\iota(e_j) \nabla_{e_j}) \]

\[ = -\omega^i \wedge \iota(e_j) \nabla_{e_i} \nabla_{e_j}. \]

So

\[ \Delta = d\delta + \delta d = -\sum_{i=1}^{m} \nabla_{e_i e_i}^2 - \sum_{i,j} \omega^i \wedge \iota(e_j) R(e_i, e_j). \]

This proves the statement.

Remark: On functions, i.e. 0-form \( f \), we have

\[ R(e_i, e_j) f = f R(e_i, e_j) 1 = 0. \]

Hence, we have, for a local orthonormal frame field,

\[ \Delta f = -\sum_{i=1}^{m} \nabla_{e_i e_i}^2 f. \]

**Theorem.** For any smooth differential form \( \eta \) \((r\text{-forms})\),

\[ -\Delta |\eta|^2 = 2|\nabla \eta|^2 + 2 \sum \nabla_{e_i e_i}^2 \eta, \eta >, \]

where

\[ |\nabla \eta|^2 = \sum_i |\nabla_{e_i} \eta|^2. \]

11
Remark: Consider the Euclidean case $\mathbb{R}^2$ and $\eta = f$. Then

$$- \triangle |f|^2 = 2f_x f_x + 2f_y f_y + 2f_{xx} f + 2f_{yy} f = 2| \nabla f |^2 + 2 < f_{xx} + f_{yy}, f > .$$

The theorem is motivated by it, and can be derived by direct computation. We only verify it for 1–form $\eta$. We only need to verify it at every point $p \in M$. Let $\{e_1, \ldots, e_m\}$ be a local orthonormal frame fields and $\{\omega_1, \ldots, \omega_m\}$ be the coframe fields. Write

$$\eta = \sum_i \eta_i \omega^i,$$

then, at point $p$, using the formula of $\triangle$ for smooth functions,

$$- \frac{1}{2} \triangle |\eta|^2 = \sum_i < \nabla_{e^i} \nabla_{e^i} \eta, \eta > + \sum_i | \nabla_{e^i} \eta |^2 .$$

This finishes the proof.

Combining the both, we get

**Bochner’s formula.** Let $e_1, \ldots, e_m$ be a local orthonormal frame field, with the dual frame field $\omega^1, \ldots, \omega^m$. Let $\eta$ be a $r$-form on $M$. Then

$$\frac{1}{2} \triangle |\eta|^2 = < \triangle \eta, \eta > - | \nabla \eta |^2 + < \omega^i \wedge \iota(e_j) R(e_i, e_j) (\eta) , \eta > .$$

**Proof:** Choose normal coordinate. From above,

$$\frac{1}{2} \triangle |\eta|^2 = - | \nabla \eta |^2 - \sum_i < \nabla_{e^i} \eta, \eta > .$$

And from the Weitzenbock’s formula,

$$\triangle \eta = - \sum_{i=1}^m \nabla_{e^i e^i} \eta - \sum_{i,j} \omega^i \wedge \iota(e_j) R(e_i, e_j)(\eta) .$$

Hence,

$$- \sum_{i=1}^m \nabla_{e^i e^i} \eta = \triangle (\eta) + \sum_{i,j} \omega^i \wedge \iota(e_j) R(e_i, e_j)(\eta) .$$
Substituting it into above, we get
\[
\frac{1}{2} \triangle |\eta|^2 = \langle \triangle \eta, \eta \rangle - |\nabla \eta|^2 + \langle \omega^i \wedge \iota(e_j) R(e_i, e_j) \eta, \eta \rangle.
\]

For any smooth one-form, we get
\[
\text{Corollary} \quad \text{Let } e_1, \ldots, e_m \text{ be a local orthonormal frame field, with the dual frame field } \omega^1, \ldots, \omega^m. \text{ Let } \eta \text{ be a smooth one-form, then}
\]
\[
\frac{1}{2} \triangle |\eta|^2 = \langle \triangle \eta, \eta \rangle - |\nabla \eta|^2 - r(\eta, \eta),
\]
where \(|\nabla \eta|^2 := \sum_i \langle \nabla e_i \eta, \nabla e_i \eta \rangle\), and writing \(\eta = \sum f_i \omega^i\),
\[

r(\eta, \eta) \:= \sum_{i,j} r(f_i e_i, f_j e_j) = \sum_{i,j} f_i f_j r(e_i, e_j).
\]

\textbf{Proof:} We only need to compute, for 1-form \(\eta\),
\[
\langle \eta, \omega^i \wedge \iota(e_j) R(e_i, e_j) \eta \rangle = \langle f_i \omega^j, \omega^i \wedge \iota(e_j) R(e_i, e_j) f_k \omega^k \rangle >
\]
\[
= -f_i f_k < \omega^j, \omega^i \wedge \iota(e_j) R_{kmij} \omega^m >
\]
\[
= -f_i f_k < \omega^j, R_{kijj} \omega^i >
\]
\[
= -f_i f_k R_{kijj}
\]
\[
= -r(\eta, \eta).
\]

\textbf{Theorem (Bochner)} \quad \text{Let } M \text{ be a compact Riemannian manifold. If } M \text{ has positive Ricci curvature, then } M \text{ has no nontrivial harmonic 1-form, thus,}
\[
H^1_{dR}(M, \mathbb{R}) = \{0\}.
\]

\textbf{Proof.} We integrate the formula above, and using the divergence theorem,
\[
0 = -\int_M \triangle |\omega| \eta = 2 \int_M (|\nabla |\omega|^2 + r(\omega, \omega)) \eta.
\]
By the assumption, the integrand on the right hand side is pointwise non-negative. It therefore has to vanish identically. Hence \(r(\omega, \omega) = 0\), which implies that \(\omega = 0\) since the Ricci curvature on \(M\) is positive. This proves the statement.
4 Proof of Garding’s inequality

Theorem Garding’s inequality: There exist constant $c_1, c_2 > 0$, such that for every $f \in \Lambda^*(M)$, we have

\[(\triangle f, f) \geq c_1 \|f\|^2 - c_2 \|f\|_0^2.\]

**Proof.** For every $f \in \Lambda^*(M)$, from Bochner’s formula,

\[
<\triangle f, f> = \frac{1}{2} \triangle |f|^2 + |\nabla f|^2 - <\omega^i \wedge \nu(e_j) R(e_i, e_j) f, f > \\
\geq \frac{1}{2} \triangle |f|^2 + |\nabla f|^2 - a_1 |f|^2,
\]

where $a_1$ is a constant independent of $f$. Note that the last inequality holds because $<\omega^i \wedge \nu(e_j) R(e_i, e_j) f, f >$ does not depend on the derivative(differential) of $f$ (Note: although $R(e_i, e_j)$ depends on the derivative(differential), but since $R(e_i, e_j)(\alpha f) = \alpha R(e_i, e_j) f$ for every $\alpha \in C^\infty(M)$, so $<\omega^i \wedge \nu(e_j) R(e_i, e_j) f, f >$ is a quadratic form on $\Lambda^*(M)$, its coefficients only depend on $M$, and since $M$ is compact, such constant $a_1$ exists). Taking the integration on $M$ and by definition, we get

\[(\triangle f, f) \geq c_1 \|f\|^2 - c_2 \|f\|_0^2 + \frac{1}{2} \int_M \triangle |f|^2 \eta.\]

But by Stokes theorem,

\[
\int_M \triangle |f|^2 \eta = 0.
\]

This proves the Garding’s inequality.