§5.3 Surface Theory with Differential Forms

1 Differential forms on $\mathbb{R}^n$, Click here to see more details

Differential forms provide an approach to multivariable calculus (Click here to see more details) that is independent of coordinates.

Let $U$ be an open set in $\mathbb{R}^n$. A differential 0-form ("zero form") is defined to be a smooth function $f$ (here smooth means that $f$ is differentiable at any order) on $U$.

If $v \in \mathbb{R}^n$, then $f$ has a directional derivative $D_v f$ (see “introduction to differentiable function” in Chapter one), which is another function on $U$ whose value at a point $p \in U$ is the rate of change (at $p$) of $f$ in the $v$ direction:

$$D_v f(p) = \frac{d}{dt} f(p + tv) \bigg|_{t=0}.$$ 

In particular, if $v = e_j$, where $\{e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, \ldots, 0), \ldots, e_n = (0, \ldots, 1)\}$ is the standard basis of $\mathbb{R}^n$, then

$$D_{e_j} f = \frac{\partial f}{\partial x_j},$$

is the partial derivative of $f$ with respect to the $j$th coordinate function, where $x_1, x_2, \ldots x_n$ are the coordinate functions on $U$. By their very definition, partial derivatives depend upon the choice of coordinates: if new coordinates $y_1, y_2, \ldots y_n$ are introduced, then

$$\frac{\partial f}{\partial x^j} = \sum_{i=1}^{n} \frac{\partial y^i}{\partial x^j} \frac{\partial f}{\partial y^i}.$$ 

The first idea leading to differential forms is the observation that $D_v f(p)$ is a linear function of $v$:

$$D_{v+w} f(p) = D_v f(p) + D_w f(p), \quad D_{cv} f(p) = c D_v f(p)$$

for any vectors $v, w$ and any real number $c$. This linear map from $\mathbb{R}^n$ to $\mathbb{R}$ is denoted $df_p$ and called the (exterior) derivative of $f$ at $p$. Thus $df_p(v) = D_v f(p)$. The object $df$ can be viewed as a function on $U$, whose value at $p$ is not a real number, but the linear map $df_p$, in other words, $df$ assigns, at every point $p \in U$, a linear
map $df_p$ from $\mathbb{R}^n$ to $\mathbb{R}$. This is just a special case of (general) differential $1$-forms. Since any vector $v$ is a linear combination $\sum v_j e_j$ of its components, $df$ is uniquely determined by $df_p(e_j)$ for each $j$ and each $p \in U$ (i.e. $df_p(v) = \sum v_j df_p(e_j)$) which are just the partial derivatives of $f$ on $U$. Thus $df$ provides a way of encoding the partial derivatives of $f$. It can be decoded by noticing that the coordinates $x_1, x_2, \ldots x_n$ are themselves functions on $U$ ($x_j$ maps each point in $U$ the $j$-th coordinate), and so define differential $1$-forms $dx_1, dx_2, \ldots dx_n$ (most time, we also write it as $dx^1, dx^2, \ldots dx^n$), in other words,

\begin{equation}
(5.1.1) \quad dx^j(e_i) = \delta_{ij}, \quad ordx^j(v) = v_j, \quad \text{for } v = (v_1, \ldots, v_n) \in \mathbb{R}^n,
\end{equation}

In general, for any $0$-form $f$,

\begin{equation}
(5.1.2) \quad df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i.
\end{equation}

The meaning of this expression is given by evaluating both sides at an arbitrary point $p$: on the right hand side, the sum is defined "pointwise", so that

$$df_p = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p) dx^i|_p.$$ 

Applying both sides to $e_j$, the result on each side is the $j$-th partial derivative of $f$ at $p$ (note: $dx^i|_p$ is actually independent of $p$, i.e. $dx^i|_p = dx^i$).

**Example.** Let $f = e^{x^2+y}$, then

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 2xe^{x^2+y}dx + e^{x^2+y}dy.$$

We now give the general definition of $1$–forms on $U \subset \mathbb{R}^n$:

**Definition 5.1.1** A $1$–form $\phi$ on $U \subset \mathbb{R}^n$ (either $n = 2$ or $n = 3$) assigns, for every $p \in U \subset \mathbb{R}^n$, a is a linear map $\phi|_p : \mathbb{R}^n \rightarrow \mathbb{R}$.

**Remarks:**

(1) In linear algebra, given a vector space $V$, the set of all linear maps $f : V \rightarrow \mathbb{R}$ is called the dual space of $V$. So the alternative definition of $1$–form $\phi$ is that the $1$–form $\phi$ on $U \subset \mathbb{R}^n$ (either $n = 2$ or $n = 3$) assigns, for every $p \in U \subset \mathbb{R}^n$, an element $\phi_p \in \mathbb{R}^{n*}$, where $\mathbb{R}^{n*}$ is the dual space of $\mathbb{R}^n$. 

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(2) In particular, the 1–forms $dx^1,\ldots, dx^n$ are defined by the property that for (see (5.1.1)) any vector $v = (v_1,\ldots, v_n) \in \mathbb{R}^n$,

$$dx^i|_p(v) = v_i.$$  

The $dx^i$ form a basis for the space of all 1–forms on $\mathbb{R}^n$, so any 1–form $\phi$ on $U \subset \mathbb{R}^n$ may be expressed in the form

$$\phi = \sum_{i=1}^{n} \phi_i dx^i, \tag{5.1.3}$$

where $\phi_1,\ldots, \phi_n$ are functions on $U$. If $v = (v_1,\ldots, v_n) \in \mathbb{R}^n$, then

$$\phi_p(v) = \sum_{i=1}^{n} f_i(p)v_i.$$  

(3) Note that $\{dx^1,\ldots, dx^n\}$ is in fact the standard basis of the dual space $\mathbb{R}^{n*}$, it is dual to the standard basis $\{e_1 = (1,0,\ldots,0), e_2 = (0,1,\ldots,0), \ldots, e_n = (0,\ldots,1)\}$ of $\mathbb{R}^n$.

**Example.** $\phi = \sin x dx + y^2 dy$ is a 1-form on $\mathbb{R}^2$.

You might ask the following question: *given a differential 1-form $\phi$ on $U$, when does there exist a function $f$ on $U$ such that $\phi = df$?* By comparing (5.1.2) and (5.1.3) reduces this question to the search for a function $f$ whose partial derivatives $\partial f/\partial x_i$ are equal to $n$ given functions $\phi_i$. For $n > 1$, such a function does not always exist: any smooth function $f$ satisfies

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i},$$

so it will be impossible to find such an $f$ unless

$$\frac{\partial \phi_j}{\partial x^i} - \frac{\partial \phi_i}{\partial x^j} = 0.$$

for all $i$ and $j$.

**Differential 2–forms on $\mathbb{R}^n$.**

For two 1–forms $\phi, \psi$, define the *wedge product* $\phi \wedge \psi$ as follows, for $p \in \mathbb{R}^n$, $\phi|_p \wedge \psi|_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is given by, for every $(v, w) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\phi|_p \wedge \psi|_p(v, w) = \phi(v)\psi(w) - \psi(v)\phi(w). \tag{5.1.4}$$
The wedge product is a way to produce 2-forms from the given two 1-forms.

By the definition, we have \( \phi \wedge \psi = -\psi \wedge \phi \), and \( \phi \wedge \phi = 0 \).

A basis for the 2-forms on \( \mathbb{R}^n \) is given by the set
\[
\{dx^{i_1} \wedge dx^{i_2} : 1 \leq i_1 < i_2 \leq n\}.
\]

Any 2-forms \( \Omega \) on \( U \) can be expressed in the form
\[
\Omega = \sum_{i_1 < i_2} f_{i_1, i_2} \, dx^{i_1} \wedge dx^{i_2},
\]
where \( f_{i_1, i_2} \) are functions on \( U \). In an alternative definition,

**Definition 5.1.2** A 2-form \( \Omega \) on \( U \subset \mathbb{R}^n \) assigns, at each point \( p \in U \), an alternating bilinear mapping \( \Omega|_p : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \). Here, “alternating” means that \( \Omega|_p(v, w) = -\Omega|_p(w, v) \), and “bilinear” means that \( \Omega|_p \) is linear in both \( v \) and \( w \). We can also define the concept of the \( k \)-form in a similar way as
\[
\sum_{i_1 < i_2 < \cdots < i_k} f_{i_1, i_2, \ldots, i_k} \, dx^{i_1} \wedge \cdots \wedge dx^{i_k}.
\]

The exterior derivative takes that takes \( k \)-forms to \((k+1)\)-forms. It is defined in a similar way as above. For example, if \( \phi = \sum_{|I|=k} f_I \, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \), then the **exterior derivative** \( d\phi \) of \( \phi \) is the \((k+1)\)-form which is given by
\[
d\phi = \sum_{|I|=k} df_I \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.
\]

If \( \phi \) is a \( p \)-form and \( \psi \) is a \( q \)-form, then the Leibniz rule takes the form
\[
d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^p \phi \wedge d\psi.
\]

**Example.** Let \( \omega = f \, dx + g \, dy + h \, dz \) be a 1-form on \( U \subset \mathbb{R}^3 \), then
\[
d\omega = df \wedge dx + dg \wedge dy + dh \wedge dz = (g_x - f_y) \, dx \wedge dy + (h_y - g_z) \, dy \wedge dz + (f_z - h_x) \, dz \wedge dx.
\]

**Very Important Theorem:** \( d^2 = 0 \).

**Proof.** We only check for 0-forms, i.e., for functions. In fact, by definition,
\[
d(df) = d \left( \sum_{j=1}^{n} \frac{\partial f}{\partial x^j} \, dx^j \right) = \sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j} \, dx^j \wedge dx^i.
\]
\[
\sum_{i<j} \left( \frac{\partial^2 f}{\partial x^j \partial x^i} - \frac{\partial^2 f}{\partial x^i \partial x^j} \right) dx^j \wedge dx^i = 0.
\]

This verifies the functions case. The general result follows from the similar method.

**pull-backs**: Given a map \( F = (x_1, x_2, x_3) : U \subset \mathbb{R}^2 \to \mathbb{R}^3 \), and \( \omega = f \, dx + g \, dy + h \, dx \) be a 1-form on \( U \subset \mathbb{R}^3 \), then we can define \( F^* \omega \), the pull-back of \( \omega \) by \( F \) as

\[
F^* \omega = f \circ F \, dx_1 + g \circ F \, dx_2 + h \circ F \, dx_3
\]

\[
= f \circ F \left( \frac{\partial x_1}{du} \, du + \frac{\partial x_1}{dv} \, dv \right) + g \circ F \left( \frac{\partial x_2}{du} \, du + \frac{\partial x_2}{dv} \, dv \right) + h \circ F \left( \frac{\partial x_3}{du} \, du + \frac{\partial x_3}{dv} \, dv \right),
\]

where \((u, v)\) is the coordinate system on \( \mathbb{R}^2 \). Note that \( F^* \omega \) is a differential form on \( U \subset \mathbb{R}^2 \), and it is easy to check that (do it by yourself)

\[
F^* d\omega = dF^* \omega.
\]

## 2 Differential forms on surfaces

We now define the concept of the differential forms on the surface \( M \) in \( \mathbb{R}^3 \). To do so, we need to look at the tangent space \( T_p(M) \). Recall that when \( M \) is flat, i.e. \( M = U \) or \( M = \mathbb{R}^3 \), then its tangent space \( T_pM \) is the whole space \( \mathbb{R}^2 \), i.e. \( T_pU \cong \mathbb{R}^2 \) for every point \( p \in U \). So when we look at the definition of 1-forms \( \phi \) on \( U \subset \mathbb{R}^3 \), actually we read it as follows: a 1-form \( \phi \) on \( U \subset \mathbb{R}^3 \) assigns, at each point \( p \in U \), a linear map \( \omega|_p : T_p(U) \cong \mathbb{R}^2 \to \mathbb{R} \), i.e., \( \phi|_p \in T^*_p(U) \cong \mathbb{R}^2^* \).

We now give the general definition of differential forms on \( M \).

**Definition 5.2.2** Given a surface \( M \) in \( \mathbb{R}^3 \), a (differential) 1–form \( \omega \) on \( M \) assigns, at every \( p \in M \), a linear map \( \omega|_p : T_p(M) \to \mathbb{R} \), i.e., \( \omega|_p \in T^*_p(M) \).

A (differential) 2–form \( \Omega \) on \( M \) assigns, at every point \( p \in M \), a two form \( \Omega|_p \in \wedge^2 T^*_p(M) \).

Every \( k \)–form on \( M \) is always zero for \( k \geq 3 \).

For a smooth function \( f \) on \( M \), the **exterior** derivative of \( f \) is the 1–form \( df \) with the property that for any \( p \in M \), \( v_p \in T_p(M) \),

\[
(5.2.1) \quad df_p(v_p) = D_v(f)(p),
\]
where \( D_v(f)(p) \) is the directional derivative of \( f \) at \( p \) with respect to the direction \( v_p \) (see (3.4.1) for the definition of the directional derivative). So as we indicated before, \( df \) provides a way of encoding all the directional derivative of \( f \) at \( p \).

Let \( \sigma : U \subset \mathbb{R}^2 \to \mathbb{R}^3 \) be a parametrization of \( M \). Then the coordinates \( u, v \) on \( \mathbb{R}^2 \) be also regarded as functions on \( M \) sending the point \( \sigma(u, v) \) to \( u \) and \( v \) respectively. Hence, \( du \) and \( dv \) are well-defined 1-forms on \( M \) (more precisely on \( \sigma(U) \)). It can be easily checked by definition that

\[
(5.2.2) \quad du\big|_p(\sigma_u|_p) = 1, \quad dv\big|_p(\sigma_u|_p) = 0, \quad dv\big|_p(\sigma_v|_p) = 0, \quad dv\big|_p(\sigma_v|_p) = 1.
\]

Hence \( \{du, dv\} \) is dual to \( \{\sigma_u, \sigma_v\} \).

For any smooth function \( f \) on \( M \), in terms of parametrization \( \sigma : U \subset \mathbb{R}^2 \to \mathbb{R}^3 \) of \( M \), we can write

\[
(5.2.3) \quad df = D\sigma_u f du + D\sigma_v f dv
\]

We we write \( f(u, v) = f \circ \sigma(u, v) \), since \( D\sigma_u f(p) = (\partial f/\partial u)(u_0, v_0) \), \( D\sigma_v f(p) = (\partial f/\partial v)(u_0, v_0) \), where \( \sigma(u_0, v_0) = p \), we have

\[
(5.2.4) \quad df = f_u du + f_v dv
\]

where \( f_u = \partial f/\partial u, f_v = \partial f/\partial v \). More precisely, we should write \((5.2.4)\) as \( \sigma^*df = f_u du + f_v dv \), but we simply write it as in \((5.2.4)\) without the danger of confusion. In other words, if we work everything on \( \mathbb{R}^2 \) by the pulling back through \( \sigma \), then we can regard \( du, dv \) as the standard 1-forms on \( \mathbb{R}^2 \) which is dual to \( \{(1,0), (0,1)\} \).

In terms of the local parametrization \( \sigma : U \subset \mathbb{R}^2 \to \mathbb{R}^3 \) of \( M \), any 1-form \( \omega \) on \( M \) can be written as

\[
\omega = adu + bdv
\]

where \( a, b \) are functions on \( \sigma(U) \). Similarly, every 2-form \( \Omega \) can be locally (on \( \sigma(U) \)) written as

\[
\Omega = Adu \wedge dv,
\]

where \( A \) is a function on \( \sigma(U) \).

For every 1-form \( \omega \) on \( M \), the exterior derivative of \( \omega \) is a 2-form, which is defined in a similar way as in the previous section, by \( \omega = adu + bdv \), then \( d\omega = da \wedge du + db \wedge dv \). (you can check that it is independent of the choice of the choices of parametrizations). It is easy to check that \( d^2 = 0 \). The exterior operator \( d \) has the following important property: \( d^2 = 0 \), i.e. for every function \( f \) on \( M \), \( d(d(f)) = 0 \).
3 The method of moving frames for curves

Let \( \mathbf{x}(t) = (x_1(t), x_2(t), x_3(t)) \) be a space curve. Let \( \mathbf{e}_1 \) be the unit-tangent vector. Let \( \mathbf{e}_2 \) such that \( \frac{d\mathbf{e}_1}{ds} = \kappa \mathbf{e}_2 \). \( \mathbf{e}_2 \) is called the principal normal and \( \kappa \) is the curvature. Let \( \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2 \) is the binormal. Then \( \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \) form an orthonormal basis (Frenet frame). Write

\[
\frac{d\mathbf{e}_i}{ds} = \sum_{j=1}^{3} \omega_{ij} \mathbf{e}_j,
\]

where \( \omega_{ij} \) are 1-forms. Since \( \langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij} \), form \( d < \mathbf{e}_i, \mathbf{e}_j > = 0 \), we have \( \omega_{ij} = -\omega_{ji} \). Hence the matrix \( (\omega)_{ij} \) is a \( 3 \times 3 \) skew-symmetric matrix, whose entries are differential 1-forms. From the skew-symmetric, we have \( \omega_{jj} = 0 \). From the selection of \( \mathbf{e}_1, \mathbf{e}_2 \) and \( \mathbf{e}_3 \), we have \( \omega_{13} = \omega_{31} = 0 \). Hence, we have (Frenet formula):

\[
\frac{d\mathbf{e}_1}{ds} = \kappa \mathbf{e}_2
\]
\[
\frac{d\mathbf{e}_2}{ds} = -\kappa \mathbf{e}_1 + \tau \mathbf{e}_3
\]
\[
\frac{d\mathbf{e}_3}{ds} = -\tau \mathbf{e}_2.
\]

4 The method of moving frames for surfaces

1. Structure equations

I suggest you to read the appendix below before you start this section.

Let \( M \) be a surface and \( \mathbf{\sigma} : U \rightarrow \mathbb{R}^3 \) be a local parametrization of \( M \). Recall that the vectors \( \{\mathbf{\sigma}_u|_p, \mathbf{\sigma}_v|_p\} \) is a basis for \( T_p(M) \). Let \( \mathbf{e}_1(p), \mathbf{e}_2(p) \) be an orthonormal basis for \( T_p(M), p \in U \) (such orthonormal basis always exists by applying Gram-Schmidt orthonormalization procedure) and let \( \mathbf{e}_3(p) = \mathbf{n}(p) \) be the unit normal (Gauss map). The key point of this section is that we are working on the orthonormal basis, NOT just the basis \( \{\mathbf{\sigma}_u|_p, \mathbf{\sigma}_v|_p\} \). The basis \( \{\mathbf{e}_1(p), \mathbf{e}_2(p), \mathbf{e}_3(p)\} \) serves as a moving frame for \( \mathbb{R}^3 \), where \( \mathbf{e}_1(p), \mathbf{e}_2(p) \) is an orthonormal basis for \( T_p(M) \).

Let \( \mathbf{\sigma} : U \rightarrow \mathbb{R}^3 \) be a local parametrization of \( M \). Consider \( d\mathbf{\sigma} = \mathbf{\sigma}_u du + \mathbf{\sigma}_v dv \). Since \( \{\mathbf{e}_1(p), \mathbf{e}_2(p)\} \) is a basis for \( T_p(M) \), we write

\[
(5.3.1) \quad \mathbf{\sigma}_u = \lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2, \quad \mathbf{\sigma}_v = \lambda_3 \mathbf{e}_1 + \lambda_4 \mathbf{e}_4.
\]
Hence we can rewrite $d\sigma$ as

\[(5.3.2) \quad d\sigma = \omega_1 e_1 + \omega_2 e_2,\]

where $\omega_1, \omega_2$ are differential 1-forms on $M$, $\omega_1 = \lambda_1 du + \lambda_3 dv$, $\omega_1 = \lambda_2 du + \lambda_4 dv$. We claim that $\{\omega_1, \omega_2\}$ is dual to $\{e_1, e_2\}$. To prove the claim, we first note that, from (5.3.1) $d\sigma(\sigma_u) = \sigma_u$, and $d\sigma(\sigma_v) = \sigma_v$, so $d\sigma(v) = v$ for every tangent vector. Then the claim can be derived by using (5.3.2) and $d\sigma(e_1) = e_1$ and $d\sigma(e_2) = e_2$. The differential 1-forms $\omega_1, \omega_2$ keep track of how our point moves around on $M$.

Next we want to see how the frame itself twists, we will define 1-forms $\omega_{ij}, i, j = 1, 2, 3$. Consider $de_i$, the exterior derivative of $e_i$. Note that $de_i$ is a vector-valued (the image is in $\mathbb{R}^3$) differential 1-form, and since $\{e_1, e_2, e_3\}$ is linearly independent, so it is a basis for $\mathbb{R}^3$. Therefore we can

\[(5.3.4) \quad de_i = \sum_{j=1}^{3} \omega_{ij} e_j,\]

where $\omega_{ij}, i, j = 1, 2, 3$ are differential 1-forms. There are total nine of such 1-forms. First we claim that $\omega_{ij} = -\omega_{ji}$. In fact, since $e_i \cdot e_j = \delta_{ij}$, by differentiating, we have, $de_i \cdot e_j + e_i \cdot de_j = 0$. This implies that $\omega_{ij} = -\omega_{ji}$. Hence, we have only three “meaningful” 1-forms $\omega_{13}, \omega_{23}$ and $\omega_{12}$. Others are just zero. If $v_p \in T_p(M)$, then $\omega_{ij}|_p(v_p)$ tells us how fast $e_i$ is twisting towards $e_j$ at $p$ as we move with velocity $v_p$. Together with the above two 1-forms above, we obtain, in total, five 1-forms: $\omega_1, \omega_2, \omega_{13}, \omega_{23}, \omega_{12}$.

To summarize, we have the following Equations for moving frame

\[(5.3.5) \quad d\sigma = \omega_1 e_1 + \omega_2 e_2,\]

\[(5.3.6) \quad de_1 = \omega_{12} e_2 + \omega_{13} e_3,\]

\[(5.3.7) \quad de_2 = \omega_{21} e_1 + \omega_{23} e_3,\]

\[(5.3.8) \quad de_3 = \omega_{31} e_1 + \omega_{32} e_2,\]

where $\omega_{ij} = -\omega_{ji}$.

Recall that the shape operator (see section 3.3) is $S_p = -dn = -de_3$, so the shape operator is embodied in the equation

\[de_3 = \omega_{31} e_1 + \omega_{32} e_2 = -(\omega_{13} e_1 + \omega_{23} e_2).\]
We now calculate the matrix of the shape operator $S_p$ respect to the basis $\{e_1, e_2\}$. In fact,

$$S_p(e_1) = -de_3(e_1) = -\omega_{31}(e_1)e_1 - \omega_{32}(e_1)e_2 = \omega_{13}(e_1)e_1 + \omega_{23}(e_1)e_2,$$

$$S_p(e_2) = -de_3(e_2) = \omega_{13}(e_2)e_1 + \omega_{23}(e_2)e_2.$$

Thus, by the definition, the matrix of the shape operator $S_p$ respect to the basis $\{e_1, e_2\}$ is

$$\begin{pmatrix}
\omega_{13}(e_1) & \omega_{13}(e_2) \\
\omega_{23}(e_1) & \omega_{23}(e_2)
\end{pmatrix}.$$

Recall that the **Gauss curvature** is the determinant of the matrix of $S_p$ (see chapter 4), so we get

$$K = \det(S_p) = \omega_{13}(e_1)\omega_{23}(e_2) - \omega_{13}(e_2)\omega_{23}(e_1) = (\omega_{13} \wedge \omega_{23})(e_1, e_2).$$

Since $\omega_{13} \wedge \omega_{23}$ is a 2-form, and the vector space of the two forms has dimension one (as we noted earlier) we can write

$$\omega_{13} \wedge \omega_{23} = \lambda \omega_1 \wedge \omega_2.$$

So from above,

$$K = (\omega_{13} \wedge \omega_{23})(e_1, e_2) = \lambda \omega_1 \wedge \omega_2(e_1, e_2) = \lambda.$$

So we have a **very important** expression for the Gauss curvature (5.3.9).

$$\omega_{13} \wedge \omega_{23} = K \omega_1 \wedge \omega_2.$$

Most of our results will come from the following:

**Theorem 5.2.1 (Structure Equations):**

(5.3.10) \hspace{1cm} d\omega_1 = \omega_{12} \wedge \omega_2; \hspace{1cm} d\omega_2 = \omega_1 \wedge \omega_{12};

and

(5.3.11) \hspace{1cm} d\omega_{ij} = \sum_{k=1}^{3} \omega_{ik} \wedge \omega_{kj}.

**Proof:** Use the property that $d^2 = 0$ and using (5.3.5), we have

$$0 = d(d\sigma) = d(\omega_1 e_1) + d(\omega_2 e_2).$$
This derives the first two equations by using (5.3.6) and (5.3.7). Also, using (5.3.6), (5.2.7), (5.3.8) and \(d^2 = 0\), we get

\[
d(d\mathbf{e}_i) = d\omega_{ij} - \omega_{ij} \wedge d\mathbf{e}_j = 0.
\]

This derives the second equation.

The structure equations give:

**Gauss Equation:** \(d\omega_{12} = -\omega_{13} \wedge \omega_{23}\),

**Mainardi-Codazzi Equation:** \(d\omega_{13} = \omega_{12} \wedge \omega_{23}; \quad d\omega_{23} = -\omega_{12} \wedge \omega_{13}\).

From (5.3.9), we have \(\omega_{13} \wedge \omega_{23} = K \omega_1 \wedge \omega_2\).

Hence we get \(d\omega_{12} = -K \omega_1 \wedge \omega_2\).

This provides a **new** and simple proof of Gauss's theorem egregium (see section 5.1), since from above, we see that \(K\) only depends on \(\omega_1\) and \(\omega_2\), so the Gauss curvature is an **intrinsic** quantity!

**Example** Under the \(\sigma : U \rightarrow \mathbb{R}^3\) be a local parametrization of \(M\) with \(E = G = \frac{4}{1 + u^2 + v^2}, F = 0\) (for example, on the \(\mathbb{R}^2\) with the metric \(g = \frac{4}{(1 + x^2 + y^2)}(dx^2 + dy^2)\)). Calculate the Gauss curvature.

**Solution.** Write \(A = 1 + u^2 + v^2\). Then \(\mathbf{e}_1 = \frac{4}{2} \mathbf{\sigma}_u, \quad \mathbf{e}_2 = \frac{4}{2} \mathbf{\sigma}_v\). Since \(\{\omega_1, \omega_2\}\) is dual to \(\{\mathbf{e}_1, \mathbf{e}_2\}\), we have

\[
\omega_1 = \frac{2}{A} du, \quad \omega_2 = \frac{2}{A} dv.
\]

To calculate the Gauss curvature, we need to find out the connection form \(\omega_{12}\). We use the structure equations \(d\omega_1 = \omega_{12} \wedge \omega_2, d\omega_2 = \omega_1 \wedge \omega_{12}\) to find out \(\omega_{12}\). From \(\omega_1 = \frac{2}{A} du\), we have

\[
d\omega_1 = d\left(\frac{2}{A}\right) \wedge du = 2 \frac{-dA}{A^2} \wedge du = \frac{4v}{A^2} du \wedge dv.
\]

Writing \(\omega_{12} = adu + bdv\), and noting that \(\omega_2 = \frac{2v}{A} dv\), from \(d\omega_1 = \omega_{12} \wedge \omega_2\) we find out that \(a = 2y/A\). Similarly, we can get \(b = -2u/A\). Hence

\[
\omega_{12} = \frac{2v}{A} du - \frac{2v}{A} dv = v\omega_1 - u\omega_2.
\]
Now we use \( d\omega_{12} = K\omega_1 \wedge \omega_2 \) to find out \( K \). Since
\[
d\omega_{12} = dv \wedge \omega_1 + v d\omega_1 - du \wedge \omega_2 - ud\omega_2
= \frac{A}{2} dv \wedge \omega_1 + \frac{4v^2}{A^2} du \wedge dv - \frac{A}{2} du \wedge \omega_2 + \frac{4u^2}{A^2} du \wedge dv
= -\frac{A}{2} \omega_1 \wedge \omega_2 + (u^2 + v^2) \omega_1 \wedge \omega_2 = -\omega_1 \wedge \omega_2.
\]
Hence \( K \equiv -1 \).

2. Further look of Gauss equation and the Codazzi equations.

To see why the equation \( d\omega_{12} = -\omega_{13} \wedge \omega_{23} \) is the same as the Gauss equation we derived before (in section 5.1), we consider an orthogonal parametrization, i.e. \( F = 0 \). In this case, \( e_1 = e_u/\sqrt{E}, e_2 = e_v/\sqrt{G} \). Then, \( \omega_1 = \sqrt{E} du, \omega_2 = \sqrt{G} dv \) (since \( \{\omega_1, \omega_2\} \) is the dual basis to \( e_1, e_2 \)). We now calculate \( \omega_{12} \). Write \( \omega_{12} = A du + B dv \), we need to determine \( A \) and \( B \). From the structure equation,
\[
-\omega_2 \wedge \omega_{12} = d\omega_1 = -\sqrt{E}_v du \wedge dv.
\]
Also
\[
-\omega_2 \wedge \omega_{12} = \sqrt{G} dv \wedge (A du + B dv) = \sqrt{G} A du \wedge dv.
\]
Hence
\[
\sqrt{G} A = -\sqrt{E}_v.
\]
This implies that
\[
A = -\frac{\sqrt{E}_v}{\sqrt{G}}.
\]
Similarly, we get
\[
B = \frac{\sqrt{G}_u}{\sqrt{E}}.
\]
Hence
\[
\omega_{12} = -\omega_{21} = -\left( \frac{\sqrt{E}_v}{\sqrt{G}} du + \frac{\sqrt{G}_u}{\sqrt{E}} dv \right).
\]
\[
d\omega_{12} = \left[ \left( \frac{\sqrt{E}_v}{\sqrt{G}} \right)_v + \left( \frac{\sqrt{G}_u}{\sqrt{E}} \right)_u \right] du \wedge dv.
\]
On the other hand, from a direct computation, we can get
\[
\omega_{13} = (de_1) \cdot e_3 = \frac{e}{\sqrt{E}} du + \frac{f}{\sqrt{E}} dv,
\]
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\[ \omega_{23} = (d\mathbf{e}_2) \cdot \mathbf{e}_3 = \frac{f}{\sqrt{G}} du + \frac{g}{\sqrt{G}} dv. \]

So

\[ \omega_{13} \land \omega_{32} = -\omega_{13} \land \omega_{23} = -eg - f^2 du \land dv. \]

Hence, Gauss equation \( d\omega_{12} = -\omega_{13} \land \omega_{23} \) is equivalent to

\[ - \left[ \left( \frac{(\sqrt{E})_v}{\sqrt{G}} \right)_v + \left( \frac{(\sqrt{G})_u}{\sqrt{E}} \right)_u \right] = \frac{eg - f^2}{\sqrt{EG}}, \]

which is the same as the Gauss equation we derived before in the case \( F = 0 \) (see section 5.1).

Similarly, we can verify the Codazzi equations listed above are the same as what we have derived earlier.

3. Normal curvature and geodesic curvature revisited.

Let \( \mathbf{e} : U \rightarrow M \) be a parametrization, and let \( C \) be a curve on \( M \) given by \( \mathbf{e}(s) = \mathbf{e}(u(s), v(s)) \) where \( s \) is the arc-length parameter. Let \( \mathbf{T}(s) = \mathbf{e}'(s) \) be the tangent vector to \( C \), and let \( \mathbf{e}_1(s) = \mathbf{T}(s), \mathbf{e}_2(s) = \mathbf{e}(s) \times \mathbf{T}(s), \mathbf{e}_3(s) = \mathbf{n}(s) \), where \( \mathbf{n} \) is the unit normal to the surface \( M \). Then, \( \{\mathbf{e}_1(s), \mathbf{e}_2(s), \mathbf{e}_3(s)\} \) is an orthonormal moving frame along \( C \) (which is called Darboux frame). Then we have

\[ \frac{d\mathbf{e}_1(s)}{ds} = \mathbf{e}_1(s), \]

\[ \frac{d\mathbf{e}_2(s)}{ds} = \kappa_g \mathbf{e}_2(s) + \kappa_n \mathbf{e}_3(s), \]

\[ \frac{d\mathbf{e}_3(s)}{ds} = -\kappa_g \mathbf{e}_1(s) + \tau_g(s) \mathbf{e}_2(s), \]

where \( \kappa_g \) is the geodesic curvature, \( \kappa_n \) is the normal curvature and \( \tau_g(s) \) is called the geodesic torsion.

Take an orthonormal moving frame (Darboux frame) \( \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \) on \( M \) with \( \mathbf{e}_3 = \mathbf{n} \), such that the restriction of this frame to the curve \( C \) is \( \{\mathbf{e}_1(s), \mathbf{e}_2(s), \mathbf{e}_3(s)\} \). We write

\[ d\mathbf{e}_1 = \omega_{12} \mathbf{e}_2 + \omega_{13} \mathbf{e}_3, \]
\[ d\vec{e}_2 = \bar{\omega}_{21}\vec{e}_1 + \bar{\omega}_{23}\vec{e}_3, \]
\[ d\vec{e}_3 = \bar{\omega}_{31}\vec{e}_1 + \bar{\omega}_{32}\vec{e}_2, \]

where \( \bar{\omega}_{ij} = -\bar{\omega}_{ji} \). We have the following theorem:

**Theorem 5.2.2** Let \( C \) be a curve on \( M \), then

\[ \kappa_g = \bar{\omega}_{12}(\vec{e}_1), \kappa_n = \bar{\omega}_{13}(\vec{e}_1), \tau_g = \bar{\omega}_{23}(\vec{e}_1). \]

**Proof:** Since \( \alpha(s) = \sigma(u(s),v(s)) \),

\[ \vec{e}_1 = T = \frac{d\alpha(s)}{ds} = \frac{du}{ds} \sigma_u + \frac{dv}{ds} \sigma_v. \]

Hence, for \( 1 \leq i,j \leq 3 \),

\[ \bar{\omega}_{ij}(T) = \frac{d\vec{e}_i}{ds}, \vec{e}_j > \]
\[ = \frac{d\vec{e}_i}{ds}(\sigma_u) + \frac{d\vec{e}_i}{ds}(\sigma_v), \vec{e}_j > \]
\[ = \frac{d\vec{e}_i}{ds}(\sigma_u, \vec{e}_j) + \frac{d\vec{e}_i}{ds}(\sigma_v, \vec{e}_j) \]
\[ = \frac{d\vec{e}_i}{ds}, \vec{e}_j >. \]

Hence

\[ \kappa_g = \bar{\omega}_{12}(T), \kappa_n = \bar{\omega}_{13}(T), \tau_g = \bar{\omega}_{23}(T). \]

This finishes the proof.

We now re-derive the formula of \( \kappa_g \) in terms of the orthogonal parameterization. Let \( \mathbf{x} : U \to S \) be a orthogonal parametrization, i.e \( F = 0 \) (where \( \{E,F,G\} \) is its first fundamental form). Let \( C \) be a curve on \( S \) and let \( \{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \) be the Darboux frame. Then, form above

\[ \kappa_g = \bar{\omega}_{12}(\vec{e}_1). \]

On the other hand, consider another (natural) orthonormal frame: i.e. let \( \vec{e}_1 = \mathbf{x}_u/\sqrt{E}, \vec{e}_2 = \mathbf{x}_v/\sqrt{G}, \vec{e}_3 = \mathbf{n} \). Then the frame \( \{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \) is also an orthonormal moving frame. Let \( \omega_1, \omega_2 \) be the dual of \( \vec{e}_1, \vec{e}_2 \) and let \( \omega_{12} \) be the connection form. Then, as we derived before,

\[ \omega_{12} = -\omega_{21} = -\frac{(\sqrt{E})_v}{\sqrt{G}}du + \frac{(\sqrt{G})_u}{\sqrt{E}}dv. \]
Next, we want to derive a relationship between $\bar{\omega}_{12}$ and $\omega_{12}$. To do so, let $\theta$ be the angle from $x_u$ to $\bar{e}_1 = T$, then we have

$$\left( \begin{array}{c} \bar{e}_1 \\ \bar{e}_2 \end{array} \right) = \left( \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right) \left( \begin{array}{c} e_1 \\ e_2 \end{array} \right).$$

Since $\{\omega_1, \omega_2\}$ is the dual basis of $\{e_1, e_2\}$ (resp., $\{\bar{\omega}_1, \bar{\omega}_2\}$ is the dual basis of $\{\bar{e}_1, \bar{e}_2\}$), hence

$$\bar{\omega}_1 = \cos \theta \omega_1 + \sin \theta \omega_2$$
$$\bar{\omega}_2 = -\sin \theta \omega_1 + \cos \theta \omega_2$$
$$\bar{\omega}_{12} = \omega_{12} + d\theta.$$

Hence

$$\bar{\omega}_{12} = \omega_{12} + d\theta = -\frac{(\sqrt{E})_v}{\sqrt{G}} du + \frac{(\sqrt{G})_u}{\sqrt{E}} dv + d\theta.$$

Since

$$e_1 = T = \frac{d\alpha(s)}{ds} = \frac{du}{ds} x_u + \frac{dv}{ds} x_v,$$

we have

$$\kappa_g = \bar{\omega}_{12}(\bar{e}_1) = -\frac{(\sqrt{E})_v}{\sqrt{G}} du(T) + \frac{(\sqrt{G})_u}{\sqrt{E}} dv(T) + d\theta(T) = -\frac{(\sqrt{E})_v}{\sqrt{G}} \frac{du(s)}{ds} + \frac{(\sqrt{G})_u}{\sqrt{E}} \frac{dv(s)}{ds} + d\theta(T).$$

Now $d\theta = \theta_udu + \theta_v dv$, so

$$d\theta(T) = \theta_u \frac{du(s)}{ds} + \theta_v \frac{dv(s)}{ds} = \frac{d\theta}{ds}.$$ 

Hence

$$\kappa_g = -\frac{(\sqrt{E})_v}{\sqrt{G}} \frac{du(s)}{ds} + \frac{(\sqrt{G})_u}{\sqrt{E}} \frac{dv(s)}{ds} + \frac{d\theta}{ds}.$$ 

Therefore we re-proved the following Liouville theorem:

**Theorem 5.2.3** Let $x : U \to S$ be an orthogonal parametrization, i.e $F = 0$ (where $\{E, F, G\}$ is its first fundamental form). Let $C$ be a curve on $S$. Then

$$\kappa_g = -\frac{(\sqrt{E})_v}{\sqrt{G}} \frac{du(s)}{ds} + \frac{(\sqrt{G})_u}{\sqrt{E}} \frac{dv(s)}{ds} + \frac{d\theta}{ds}.$$ 

6. **A simple proof of Gauss-Bonnet theorem.** The formula

$$d\omega_{12} = -K \omega_1 \wedge \omega_2$$
is the key in applying Green’s theorem to prove Gauss-Bonnet theorem. The formula can be re-written as

\[ K d\sigma = -d\omega_{12}, \]

where \( d\sigma = \omega_1 \wedge \omega_2 \).

From the Gauss equation and Stoke’s Theorem, the Gauss-Bonnet formula follows immediately for an oriented surface \( M \) with (piecewise smooth) boundary \( \partial M \) on which we can globally define a moving frame. That is, we can reprove the local Gauss-Bonnet formula quite effortlessly.

**Proof**: We start with an arbitrary moving frame \( e_1, e_2, e_3 \), and take a Darboux frame (i.e. a moving frame for the surface with \( e_1 \) tangent to \( \partial M \)) \( \bar{e}_1, \bar{e}_2, \bar{e}_3 \) along \( \partial M \). We write \( \bar{e}_1 = \cos \theta e_1 + \sin \theta e_2 \), \( \bar{e}_2 = -\sin \theta e_1 + \cos \theta e_2 \) (where \( \theta \) is smoothly chosen along the smooth pieces of \( \partial M \) and the exterior angle \( \epsilon_j \) at \( P_j \) gives the jump of theta as we cross \( P_j \)). Then, by Stokes’ theorem, we have

\[
\int \int_M K d\sigma = -\int \int_M d\omega_{12} = -\int_{\partial M} \omega_{12} = -\int_{\partial M} (\bar{\omega}_{12} - d\theta) = -\int_{\partial M} \kappa_g ds + (2\pi - \sum \epsilon_j).
\]

### 7. Covariant Derivative, Connection Form

The covariant derivative \( D_v e_1 \) is the tangential component of \( d e_1(v) = \omega_{12}(v)e_2 + \omega_{13}(v)e_3 \). Hence, \( D_v e_1 = pr(de_1(v)) = \omega_{12}(v)e_2 \). Similarly, \( D_v e_2 = \omega_{21}(v)e_1 \). The form \( \omega_{12} \) is called the connection form and it measures the tangential twist of \( e_1 \) and \( e_2 \).

### 5 Appendix: Review of Surface Theory

We review here the theory of the surfaces we have learnt so far. Let \( M \) be a surface and \( \sigma : U \rightarrow M \) be an orthogonal parametrization (i.e. \( F = 0 \) in the first fundamental form). Recall that the vectors \( \{\sigma_u, \sigma_v\} \) span \( T_p(M) \). From \( 0 = F = \sigma_u \cdot \sigma_v \), we see that \( \sigma_u \) and \( \sigma_v \) are orthogonal. Let \( e_1 = \frac{\sigma_u}{\sqrt{E}}, \ e_2 = \frac{\sigma_v}{\sqrt{G}} \). Then \( \{e_1(p), e_2(p)\} \) forms an orthonormal basis for \( T_p(M), p \in M \) (i.e. \( ||e_1(p)|| = 1, ||e_2(p)|| = 1 \) and \( e_1(p) \cdot e_2(p) = 0 \)). So \( \{e_1, e_2\} \) is called a moving frame for the tangent spaces of \( S \) (since for each point \( p \in M \), we have that \( \{e_1(p), e_2(p)\} \) forms an orthonormal basis for \( T_p(M) \), here the name ”moving” because \( p \) varies from \( M \) and the name ”frame” comes from the fact it is a basis). Such orthonormal basis exists because we can always apply the Gram-Schmidt orthonormalization procedure if \( \sigma \) is not an
orthogonal parametrization. Let $e_3 = n$ be the unit normal (Gauss map). Then $\{e_1, e_2, e_3\}$ forms an (moving) orthonormal basis for $\mathbb{R}^3$.

We have (for an orthogonal parametrization) (do your own calculations using the formulas in section 5.1), in below, $E, F, G$ is first fundamental form with $F = 0$ and $e, f, g$ are the second fundamental form.

$$
(e_1)_u = -\frac{E_v}{2\sqrt{EG}}e_2 + \frac{e}{\sqrt{E}}e_3,
$$

$$
(e_1)_v = \frac{G_u}{2\sqrt{EG}}e_2 + \frac{f}{\sqrt{E}}e_3,
$$

$$
(e_2)_u = \frac{E_v}{2\sqrt{EG}}e_1 + \frac{f}{\sqrt{G}}e_3,
$$

$$
(e_2)_v = \frac{G_u}{2\sqrt{EG}}e_1 + \frac{g}{\sqrt{G}}e_3,
$$

$$
(e_3)_u = -\frac{e}{\sqrt{E}}e_1 - \frac{f}{\sqrt{G}}e_2,
$$

$$
(e_3)_v = -\frac{f}{\sqrt{E}}e_1 - \frac{g}{\sqrt{G}}e_2.
$$

Gauss equation:

$$
K = -\frac{1}{2\sqrt{EG}} \left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v + \left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u.
$$

Codazzi equations:

$$
\left(\frac{e}{\sqrt{E}}\right)_v - \left(\frac{f}{\sqrt{E}}\right)_u - g \frac{(\sqrt{E})_v}{G} - f \frac{(\sqrt{G})_u}{\sqrt{EG}} = 0,
$$

$$
\left(\frac{g}{\sqrt{G}}\right)_u - \left(\frac{f}{\sqrt{G}}\right)_v - e \frac{(\sqrt{G})_u}{E} - f \frac{(\sqrt{E})_v}{\sqrt{EG}} = 0.
$$