1 Differential operators

In this notes, we always assume that $M$ is a compact oriented Riemannian manifold with the Levi-Civita connection $\nabla$, and $X, Y, ...$ are smooth vector fields.

- $(p, q)$-tensor is a smooth section of the bundle $T^p_q M = (\otimes^p T M) \otimes (\otimes^q T^* M)$.

- Contractions $c_{ij} : T^p_q M \to T^{p-1}_{q-1}$ defined by
  
  $c_{ij}(x_1 \otimes x_p \otimes y^*_1 \otimes y^*_q) = y^*_j(x_i) x_1 \otimes \cdots \hat{x}_i \otimes \cdots \otimes x_p \otimes y^*_1 \otimes \cdots \hat{y}^*_j \otimes y^*_q$.

- The interior product $i_X$ of a vector field $X$ with a covariant tensor of type $(0, p)$ is the tensor $c_{1,1}(X \otimes S)$ (of the type $(0, p-1)$), defined as
  
  $(i_X(S))_m(x_1, \ldots, x_{p-1}) = S_m(X(m), x_1, \ldots, x_{p-1})$.

- Extension of the covariant derivative: Let $X$ be a vector field on $X$. The endomorphsim $D_X$ of of $\Gamma(TM)$ has a unique extension of the endomorphism of the space of tensors, still denoted by $D_X$ which is type-preserving and satisfies the following
  
  (1) For any $S \in \Gamma(T^p_q M) \ (p > 0, q > 0)$ and any contraction $c$, $D_X(c(S)) = c(D_X S)$.
  
  (2) For any tensors $S$ and $T$,
  
  $D_X(S \otimes T) = D_X S \otimes T + S \otimes D_X T$.

  If for example $S \in \Gamma(T^0_q M)$, the $X_i$ are vector fields on $M$, then

  $$(D_XS)(X_1, \ldots, X_q) = X(S(X_1, \ldots, X_q)) - \sum_{i=1}^q S(X_1, \ldots, X_{i-1}, D_X X_i, \ldots, X_q).$$

  The readers should aware that, in general,

  $$(D_XS)(X_1, \ldots, X_n) \neq D_X(S(X_1, \ldots, X_n)).$$

- Hessian: If $f \in C^\infty(M)$, then the $(0, 2)$-tensor $Ddf$ is symmetric. The tensor $Ddf$ is called the Hessian of $f$. 

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• Musical isomorphism: \( \omega(Y) = g(X,Y) \) gives the map between \( X \) and \( \omega \). Flat \( \flat \): the map \( X \to \omega \), \( \# \) is the inverse map.

• \( \text{div}(X) := -tr(DX) \) and \( \delta \alpha = -tr D\alpha \) (the co-differential of \( \alpha \)) where \( DX \) is the endormorphism \( u \to D_u X \) and the \( D\alpha \) is a covariant two-tensor, where the trace is computed with respect to the Riemannmnian metric. Using the ”musical isomorphism”, these two notations can be viewed as equivalent. More general, for any \((0,q)\)-tensor, we define the \( \text{div}(T) \) as a \((0, q-1)\) type tensor as \( \delta T = -tr_{12} DT \) where the notation \( _{12} \) just means that the trace is taken with respect to the first two variables. It is easy to check

\[
\delta(fT) = f\delta T - i_{\nabla f} T.
\]

• The Laplacian of \( f \) is the function \( \triangle f \) given by

\[
\triangle f = \text{div} \nabla f = \delta df.
\]

The Euclidean Laplacian on \( \mathbb{R}^n \) is given by

\[
\triangle = - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.
\]

• **Bochner’s formula**: For any smooth functions on \((M, g)\), we have

\[
g(\triangle(df), df) = |Ddf|^2 + \frac{1}{2} \triangle (|df|^2) + \text{Ric}(\nabla f, \nabla f).
\]

• **The divergence operator**. Let \( X \in \Gamma(TM) \). Define

\[
\text{div}(X) = tr\{Y \to \nabla_Y X\}.
\]

\( \text{div}(X) \) is a smooth function \( M \). It is called the divergence of \( X \). The map \( \text{div} : \Gamma(TM) \to C^\infty(M) \) given by \( X \mapsto \text{div}(X) \) is called the divergence operator. In terms of local coordinate \((U; x^i)\), write

\[
X|_U = X^i \frac{\partial}{\partial x^i},
\]
then
\[ \text{div}(X) = \frac{1}{\sqrt{G}} \sum_{i=1}^{m} \frac{\partial}{\partial x^i} \left( \sqrt{G} X^i \right), \]
where \( G = \det(g_{ij}), g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j). \)

**Proof.** Let \((U, x^i)\) be a local coordinate, and write \(X|_U = X^i \frac{\partial}{\partial x^i}.\)

Then
\[ \nabla X = \left( \frac{\partial X^i}{\partial x^j} + X^k \Gamma^i_{kj} \right) dx^j \otimes \frac{\partial}{\partial x^i}. \]

Hence
\[ \text{div}(X) = \sum_{i=1}^{m} \left( \frac{\partial X^i}{\partial x^i} + X^k \Gamma^i_{ki} \right). \]

So we see that the div operator is a differential operator of first order acting on \(X\).
By the formula,

$$
\Gamma^i_{ki} = \frac{1}{2} g^{ij} \frac{\partial g_{ij}}{\partial x^k} = \frac{1}{2G} \frac{\partial G}{\partial x^k} = \frac{1}{\sqrt{G}} \frac{\partial \sqrt{G}}{\partial x^k}.
$$

Thus

$$
div(X) = \frac{\partial X^i}{\partial x^i} + \frac{X^k}{\sqrt{G}} \frac{\partial \sqrt{G}}{\partial x^k} = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i} \left( \sqrt{G} X^i \right).
$$

• The gradient of $f$. Let $f \in C^\infty(M)$, define a tangent vector field $grad(f) \in \Gamma(TM)$, by

$$
g(\text{grad}(f), X) = df(X) = X(f),
$$

for every smooth tangent vector field $X$. The tangent vector field $\text{grad}(f)$ is called the gradient of $f$. In terms of local coordinate $(U; x^i)$,

$$
\text{grad}(f) = \sum_{j=1}^m \left( \sum_{i=1}^m g^{ij} \frac{\partial f}{\partial x^i} \right) \frac{\partial}{\partial x^j},
$$

where $g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j), (g^{ij}) = (g_{ij})^{-1}$. Again, the "grad operator" $\text{grad} : C^\infty(M) \to \Gamma(TM)$ defined by $f \mapsto \text{grad}(f)$ is a differential operator of first order acting on $f$.

• Beltrami-Laplace operator. Let $f \in C^\infty(M)$, define $\Delta f = div(\text{grad}(f))$. It is called the Beltrami-Laplace operator. The operator $\Delta f = div \circ \text{grad} : C^\infty(M) \to C^\infty(M)$ is a very important differential operator.

In local coordinate $(U; x^i)$,

$$
\Delta f = \frac{1}{\sqrt{G}} \sum_{i=1}^m \frac{\partial}{\partial x^i} \left( \sum_{j=1}^m \sqrt{G} g^{ij} \frac{\partial f}{\partial x^j} \right),
$$

where $G = \det(g_{ij}), g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j), (g^{ij}) = (g_{ij})^{-1}$.

• Volume form, interior product. In a local coordinate $(U; x^i)$, let

$$
\eta = \sqrt{G} dx^1 \wedge \cdots \wedge dx^m.
$$
\( \eta \) in fact is a global \( m \)-form, called the volume form of \( M \). Fix a smooth tangent vector field \( X \), the interior product \( i(X) \) is defined by, for every tangent vector fields \( X_1, \ldots, X_{m-1} \),

\[
(i(X)\eta)(X_1, \ldots, X_{m-1}) = \eta(X, X_1, \ldots, X_{m-1}).
\]

Then we have, for every smooth tangent vector field \( X \),

\[
d(i(X)\eta) = \text{div}(X)\eta.
\]

**Proof:** By definition, \( \eta = \sqrt{G} dx^1 \wedge \cdots \wedge dx^m \). Hence, from the forumla above,

\[
div(X)\eta = \frac{\partial}{\partial x^i} \left( \sqrt{G} X^i \right) dx^1 \wedge \cdots \wedge dx^m
\]

\[
= \sum_{i=1}^m d((-1)^{i+1} \sqrt{G} X^i) \wedge dx^1 \wedge \cdots \wedge d\hat{x}^i \wedge \cdots \wedge dx^m.
\]

Write

\[
\omega = \sum_{i=1}^m (-1)^{i+1} \sqrt{G} X^i dx^1 \wedge \cdots \wedge d\hat{x}^i \wedge \cdots \wedge dx^m.
\]

It is easy to verify that \( \omega \) is independent of the choice of coordinates. So \( \omega \) is a globally defined \((m - 1)\)-form on \( M \). The above identity gives

\[
div(X)\eta = d\omega.
\]

It is easy to verify that \( \omega = i(X)\eta \) by definition.

- **The divergence theorem.** Use the above identity that \( div(X)\eta = d\omega \). and Stokes theorem, we get the divergence theorem: Let \((M, g)\) be a compact oriented Riemannian manifold, then, for every smooth tangent vector field \( X \),

\[
\int (div X)\eta = 0,
\]

where \( \eta \) is the volume form.
- **Global inner product for differential forms** We first define the inner product for differential forms. Let $\phi, \psi$ are two $r$-forms. Let $(U, x^r)$ be a local coordinate. We write

$$
\phi|_U = \frac{1}{r!} \phi_{i_1 \ldots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r},
$$

$$
\psi|_U = \frac{1}{r!} \psi_{j_1 \ldots j_r} dx^{j_1} \wedge \cdots \wedge dx^{j_r}.
$$

We define, the inner product $\langle \ , \ \rangle$ of $\phi, \psi$ as

$$
\langle \phi, \psi \rangle = \sum_{i_1 < \cdots < i_r} \phi^{i_1 \cdots i_r} \psi_{i_1 \cdots i_r},
$$

where $\phi^{i_1 \cdots i_r} = g^{i_1 j_1} \cdots g^{i_r j_r} \phi_{j_1 \cdots j_r}$. It is important to note that the definition is independent of the choice of local coordinates. We also have $\langle \phi, \phi \rangle \geq 0$ and $\langle \phi, \phi \rangle = 0$ if and only if $\phi = 0$.

We now define the **global** inner product of $\phi, \psi$ as

$$
(\phi, \psi) = \int_M \langle \phi, \psi \rangle \eta,
$$

where $\eta$ is the volume form of $M$.

- **The exterior differential operator $d$ and its co-operator** Denote by $\Lambda^r(M)$ the set of smooth $r$-forms on $M$. Let $(\ , \ )$ be the (global) inner product defined above. As the formal adjoint operator of the exterior differential operator $d$, the **codifferential operator** $\delta : \Lambda^{r+1}(M) \to \Lambda^r(M)$ is defined by, for every $\phi \in \Lambda^r(M), \psi \in \Lambda^{r+1}(M)$,

$$
(d\phi, \psi) = (\phi, \delta \psi).
$$

- **Hodge-star operator.** In order to find the expression of the codifferential operator $\delta$, we introduce the Hodge-star operator $*$, which is an isomorphism $* : \Lambda^r(M) \to \Lambda^{m-r}(M)$ defined by, for every $\phi, \eta \in \Lambda^r(M)$,

$$
\phi \wedge (*\psi) = \langle \phi, \psi \rangle \eta.
$$
Let $\omega$ be a $r$-form. Let $(U, x^i)$ be a local coordinate. We write

$$\omega|_U = \frac{1}{r!} \sum_{i_1, \ldots, i_r} a_{i_1 \ldots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r}.$$ 

Then

$$\ast \omega = \frac{\sqrt{G}}{r! (m-r)!} \delta_{i_1 \ldots i_m}^{1 \ldots m} a^{i_1 \ldots i_r} dx^{i_{r+1}} \wedge \cdots \wedge dx^{i_m},$$

where

$$a^{i_1 \ldots i_r} = g^{i_1 j_1} \cdots g^{i_r j_r} a_{j_1 \ldots j_r},$$

and $\delta_{i_1 \ldots i_m}^{1 \ldots m}$ is the Levi-Civita permutation symbol, i.e. $\delta_{i_1 \ldots i_m}^{1 \ldots m} = 1$ if $(i_1 \cdots i_m)$ is an even permutation of $(1 \ldots m)$, $\delta_{i_1 \ldots i_m}^{1 \ldots m} = -1$ if $(i_1 \cdots i_m)$ is an odd permutation of $(1 \ldots m)$, $\delta_{i_1 \ldots i_m}^{1 \ldots m} = 0$ otherwise. It can be shown that $\ast \omega$ is independent of the choice of local coordinates. So $\ast \omega$ is a globally defined $(m-r)$-form (it can be regarded as an alternative definition). The operator $\ast$ which sends $r$-forms to $(m-r)$-forms.

It has the following properties, for any $r$-forms $\phi$ and $\psi$:

1. $\phi \wedge \ast \psi = \langle \phi, \psi \rangle \eta$,
2. $\ast \eta = 1$, $\ast 1 = \eta$,
3. $\ast (\ast \phi) = (-1)^{r(m+1)} \phi$,
4. $\langle \ast \phi, \ast \psi \rangle = \langle \phi, \psi \rangle$.

- **Expression of the codifferential operator $\delta$ in terms of the Hodge-Star operator.** Define $\delta = (-1)^{mr+1} \ast d \circ \ast : \Lambda^{r+1}(M) \to \Lambda^r(M)$, where $\Lambda^r(M)$ is the set of smooth $r$-forms, is called the codifferential operator. It is easy to verify that $\delta \circ \delta = 0$. We also have the following very important property for $\delta$: For $\phi \in \Lambda^r(M)$, $\psi \in \Lambda^{r+1}(M)$, we have

$$\langle d\phi, \psi \rangle = \langle \phi, \delta \psi \rangle,$$

i.e. $\delta$ is conjugate to $d$. So $(-1)^{mr+1} \ast d \circ \ast$ is the expression of the codifferential operator $\delta$.  

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Proof. Note

\[ d(\phi \wedge *\psi) = d\phi \wedge *\psi + (-1)^r \phi \wedge d(*\psi) \]
\[ = d\phi \wedge *\psi + (-1)^r (-1)^{mr+r} \phi \wedge *(d * \psi) \]
\[ = d\phi \wedge *\psi - \phi \wedge *\delta \psi. \]

Then desired identity is obtained by applying the Stokes theorem.

- **Hodge-Laplace operator.** We define the Hodge-Laplace operator

\[ \tilde{\Delta} = d\delta + \delta d : \Lambda^r(M) \to \Lambda^r(M). \]

For \( f \in C^{\infty}(M) \), then \( \delta(f) = 0 \), so

\[ \tilde{\Delta}(f) = \delta(df) = - * d * df, \quad \tilde{\Delta} f \eta = *(\tilde{\Delta} f) = -d \ast df. \]

Let \((U, x^i)\) be a local coordinate, then

\[ df|_U = \frac{\partial f}{\partial x^i} dx^i, \]

\[ *(df)|_U = \frac{\sqrt{G}}{(m-1)!} \delta_{i_1 \cdots i_m}^{j_1 \cdots j_m} g^{i_1 j} \frac{\partial f}{\partial x^j} dx^{i_2} \wedge \cdots \wedge dx^{i_m} \]
\[ = \sqrt{G} \sum_{i=1}^m (-1)^{i+1} g^{i j} \frac{\partial f}{\partial x^j} dx^1 \wedge \cdots \wedge dx^i \wedge \cdots \wedge dx^m. \]

Hence

\[ (\tilde{\Delta} f) \eta|_U = -d(*(df))|_U = - \frac{\partial}{\partial x^i} \left( \sqrt{G} g^{ij} \frac{\partial f}{\partial x^j} \right) dx^1 \wedge \cdots \wedge dx^m \]
\[ = - \Delta f \eta|_U. \]

This tells us

\[ \tilde{\Delta} f = - \Delta f. \]

So \( -\tilde{\Delta} \) when acts on \( C^{\infty}(M) \) is the Beltrami-Laplace operator \( \Delta \).
**Hodge Theory.** In this section, we denote the Hodge-Laplace operator by $\triangle$. Let $\mathcal{H}^r(M) = \ker \triangle$ and $\mathcal{H} = \bigoplus \mathcal{H}^r(M)$. Let $\Lambda^*(M) = \bigoplus_{r=0}^{\infty} \Lambda^r(M)$.

**The Hodge theorem** Let $(M, g)$ be an $n$-dimensional compact oriented Riemannian manifold without boundary. For each integer $0 \leq r \leq n$, $\mathcal{H}^r(M)$ is finite dimensional, and there exists a bounded linear operator $G : \Lambda^*(M) \to \Lambda^*(M)$ (called Green's operator) such that

(a) $\ker G = \mathcal{H}$;

(b) $G$ keeps types, and commute with the operators $\ast, d$ and $\delta$;

(c) $G$ is a compact operator, i.e. the closure of image of an arbitrary bounded subset of $\Lambda^*(M)$ under $G$ is compact;

(d) $I = \mathcal{H} + \triangle \circ G$, where $I$ is the identity operator, and $\mathcal{H}$ is the orthogonal projection from $\Lambda^*(M)$ to $\mathcal{H}$ with respect to the inner product $(\ , )$.

From the Hodge theorem, since $I = \mathcal{H} + \triangle \circ G$, we can write (called the Hodge-decomposition)

**Corollary (Hodge-decomposition)**

\[
\Lambda^r(M) = \triangle(\Lambda^r(M)) \oplus \mathcal{H}^r(M) \\
= d\delta\Lambda^r(M) \oplus \delta d\Lambda^r(M) \oplus \mathcal{H}^r(M) \\
= d\Lambda^{r-1}(M) \oplus \delta\Lambda^{r+1}(M) \oplus \mathcal{H}^r(M).
\]