1. Let $\omega = xy \, dx + zdy - yz \, dz$, $\eta = x \, dx - y^2 \, dy - 2xz \, dz$, and let $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$f(u,v) = (uv, u^2, 3u + v), \quad (u,v) \in \mathbb{R}^2.$$ 

Find: (1) $d\omega$; (2) $d\eta$; (3) $d\omega \wedge \eta - \omega \wedge d\eta$. 

2. Define a two form on $\mathbb{R}^2$ by

$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \cdots + dx_{2n-1} \wedge dx_{2n}.$$ 

This is called the standard sympletic form on $\mathbb{R}^{2n}$. Compute $\omega^n$. 

3. Let $(r, \theta)$ be the polar coordinates defined on $\mathbb{R}^2$ outside of the origin. Write 1-forms $dr, d\theta$ in terms of the ordinary coordinates $x, y$. 

4. Define an 1-form $\omega$ on the punctured planes $\mathbb{R}^2\setminus\{0\}$ by

$$\omega = \left(\frac{-y}{x^2 + y^2}\right) dx + \left(\frac{x}{x^2 + y^2}\right) dy.$$ 

(i) Calculate $\int_C \omega$ for any circle $C$ of radius $r$ around the origin. 

(ii) Prove that $\omega$ is closed, i.e. $d\omega = 0$; 

(iii) Prove that in the half-plane $\{x > 0\}$, $\omega$ exact, i.e. $\omega$ is the differential of a function, (Hint: Try $\arctan(y/x)$). 

(iv) Prove that $\omega$ is not exact on $\mathbb{R}^2\setminus\{0\}$, i.e. there does not exist a function $f$ globally on $\mathbb{R}^2\setminus\{0\}$ such that $\omega = df$. 

5. Consider the 1-form $\omega = x_1 x_2 x_3 dx_1 + x_1^2 x_2 x_3 dx_3$ on $\mathbb{R}^3$. Verify by direct computation that $d^2 \omega = 0$. 

6. Let $M$ be a surface in $\mathbb{R}^3$ whose first fundamental form is given by $E = 1, F = 0$ and $G = u^2$. Use the moving frame method to calculate its Gauss curvature. 

7. Assume that we know the first fundamental form $E, F, G$ with $F = 0$. Derive from the structure equations $d\omega_{13} = \omega_{12} \wedge \omega_{23}$ and $d\omega_{23} = -\omega_{12} \wedge \omega_{13}$ the Mainardi-Codazzi equations.