§3-3: Integrals of Differential Forms

Now assume that (2.43) holds for exterior differential forms of degree \(< r\). We need to show that it also holds for exterior differential \(r\)-forms. Suppose \(\beta\) is a monomial of degree \(r\), and \(\beta \in A^r(N)\) is written

\[
\beta = \beta_1 \wedge \beta_2,
\]

where \(\beta_1\) is a differential 1-form on \(N\), and \(\beta_2\) is an exterior differential \((r-1)\)-form on \(N\). Then by the induction hypothesis we have

\[
d \circ f^*(\beta_1 \wedge \beta_2) = d(f^* \beta_1 \wedge f^* \beta_2)
\]

\[
= d(f^* \beta_1) \wedge f^* \beta_2 - f^* \beta_1 \wedge d(f^* \beta_2)
\]

\[
= f^*(d\beta_1 \wedge \beta_2) - f^*(\beta_1 \wedge d\beta_2)
\]

\[
= f^* \circ d(\beta_1 \wedge \beta_2).
\]

§3-3 Integrals of Differential Forms

The simplest device connecting the local and global properties of a manifold is the integration of exterior differential forms on the manifold. To define these integrals we need some prerequisites.

**Definition 3.1:** An \(m\)-dimensional smooth manifold \(M\) is called **orientable** if there exists a continuous and nonvanishing exterior differential \(m\)-form \(\omega\) on \(M\). If \(M\) is given such an \(\omega\), then \(M\) is said to be **oriented**. If two such forms are given on \(M\) such that they differ by a function factor which is always positive, then we say that they assign the same orientation to \(M\).

There exist exactly two orientations on a connected orientable manifold. The reason is that if \(\omega, \omega'\) are two exterior differential \(m\)-forms giving orientations to \(M\), then there is a nonvanishing continuous function \(f\) such that

\[
\omega' = f \omega.
\]  

Since \(M\) is connected, \(f\) retains the same sign on all of \(M\). Therefore the orientation given by \(\omega'\) is either identical to the one given by \(\omega\) or the one given by \(-\omega\).

Suppose \(M\) is oriented by the exterior differential form \(\omega\), and \((U; \{u^i\})\) is any local coordinate system on \(M\). Then \(du^1 \wedge \cdots \wedge du^m\) and \(\omega|_U\) are the same up to a non-zero factor. If the factor is positive, then we say that \((U; \{u^i\})\) is a coordinate system consistent with the orientation of \(M\). It is obvious that for any oriented manifold there exists a coordinate covering which is consistent with the orientation of the manifold, and the Jacobian of the change of coordinates...
between any two coordinate neighborhoods with nonempty intersection will always be positive. Conversely, using the partition of unity theorem below, we can prove that if there exists a compatible coordinate covering such that the Jacobian of the change of coordinates in the intersection of two neighborhoods is always positive, then $M$ is orientable. (We leave the proof to the reader.)

**Definition 3.2.** Suppose $f : M \to \mathbb{R}$ is a real function on $M$. The support of $f$ is the closure of the set of points at which $f$ is nonzero:

$$\text{supp } f = \{p \in M | f(p) \neq 0\}.$$  \hfill (3.2)

If $\phi$ is an exterior differential form, then the support of $\phi$ is

$$\text{supp } \phi = \{p \in M | \phi(p) \neq 0\}.$$  \hfill (3.3)

Obviously, the compliment of $\text{supp } \phi$ is exactly the largest open subset of $M$ on which $\phi = 0$.

**Definition 3.3.** Suppose $\Sigma_0$ is an open covering of $M$. If every compact subset of $M$ intersects only finitely many elements of $\Sigma_0$, then $\Sigma_0$ is called a **locally finite** open covering of $M$.

**Theorem 3.1.** Suppose $\Sigma$ is a topological basis of the manifold $M$. Then there is a subset $\Sigma_0$ of $\Sigma$ such that $\Sigma_0$ is a locally finite open covering of $M$.

**Proof.** By the definition of manifolds, $M$ is locally compact. Since we have assumed that $M$ satisfies the second countability axiom, there exists a countable open covering $\{U_i\}$ of $M$ such that the closure $\overline{U_i}$ of every $U_i$ is compact. Let

$$P_i = \bigcup_{1 \leq r \leq i} \overline{U_r}.$$  \hfill (3.4)

Then $P_i$ is compact, $P_i \subset P_{i+1}$, and $\bigcup_{i=1}^{\infty} P_i = M$. Now we construct another sequence of compact sets $Q_i$ satisfying

$$P_i \subset Q_i \subset \overset{0}{Q_{i+1}},$$  \hfill (3.5)

where $Q_{i+1}$ means the interior of $Q_{i+1}$.

By induction, assume that $Q_1, \ldots, Q_i$ have been constructed. Since $Q_i \cup P_{i+1}$ is compact, there exist finitely many elements $U_{\alpha}, 1 \leq \alpha \leq s$ of $\{U_j\}$ which together form a covering of $Q_i \cup P_{i+1}$. Let

$$Q_{i+1} = \bigcup_{1 \leq \alpha \leq s} \overline{U_{\alpha}}.$$  \hfill (3.6)
Then \( Q_{i+1} \) satisfies condition (3.5). First, \( Q_{i+1} \) is compact, and \( P_{i+1} \subset Q_i \). Furthermore

\[
Q_i \subset \bigcup_{1 \leq \alpha \leq s} U_\alpha \subset Q_{i+1},
\]

Obviously, \( \bigcup_{i=1}^\infty Q_i = M \).

Now let

\[
L_i = Q_i - \overset{\circ}{Q}_{i-1}, \quad K_i = \overset{\circ}{Q}_{i+1} - Q_{i-2},
\]

where \( 1 \leq i < +\infty \), and \( Q_{-1} = Q_0 = \emptyset \). (see Figure 9). Then \( L_i \) is compact, \( K_i \) is open, and \( L_i \subset K_i \).

By assumption, \( \Sigma \) is a topological basis of \( M \). Thus \( K_i \) can be expressed as a union of elements of \( \Sigma \). Since \( L_i \) is compact, and \( L_i \subset K_i \), there exist

\[
L_i \subset \bigcup_{1 \leq \alpha \leq r_i} V_{\alpha}^{(i)} \subset K_i.
\]

Because \( \bigcup_{i=1}^\infty L_i = M \),

\[
\Sigma_0 = \{ V_{\alpha}^{(i)} \mid 1 \leq \alpha \leq r_i, 1 \leq i < +\infty \}
\]

is a subcovering of \( \Sigma \).

We now show that \( \Sigma_0 \) is locally finite. Suppose \( A \) is an arbitrary compact set. By (3.4) we know that there exists a sufficiently large integer \( i \) such that \( A \subset P_i \subset Q_i \). For \( k \geq i + 2 \),

\[
K_k = \overset{\circ}{Q}_{k+1} - Q_{k-2} \subset \overset{\circ}{Q}_{k+1} - Q_i,
\]
hence $K_k \cap Q_i = \emptyset$. Therefore

$$V_{\alpha}^{(k)} \cap A \subset K_k \cap Q_i = \emptyset, \quad 1 \leq \alpha \leq \tau_k, \quad k \geq i + 2,$$

that is, only finitely many elements of $\Sigma_0$ intersect $A$.

\[\square\]

**Theorem 3.2 (Partition of Unity Theorem).** Suppose $\Sigma$ is an open covering of a smooth manifold $M$. Then there exists a family of smooth functions $\{g_\alpha\}$ on $M$ satisfying the following conditions:

1) $0 \leq g_\alpha \leq 1$, and $\text{supp } g_\alpha$ is compact for each $\alpha$. Moreover, there exists an open set $W_i \in \Sigma$ such that $\text{supp } g_\alpha \subset W_i$;

2) For each point $p \in M$, there is a neighborhood $U$ that intersects $\text{supp } g_\alpha$ for only finitely many $\alpha$;

3) $\sum_\alpha g_\alpha = 1$.

Because of condition 2), for any point $p \in M$, there are only finitely many nonzero terms on the left hand side of condition 3). Thus the summation is meaningful. The family $\{g_\alpha\}$ is called a **partition of unity** subordinate to the open covering $\Sigma$.

**Proof.** Because $M$ is a manifold, there is a topological basis $\Sigma_0 = \{U_\alpha\}$ such that each element $U_\alpha$ is a coordinate neighborhood, $\overline{U_\alpha}$ is compact, and there also exists $W_i \in \Sigma$ such that $\overline{U_\alpha} \subset W_i$. By Theorem 3.1, $\Sigma_0$ has a locally finite subcovering, so we may assume that $\Sigma_0$ itself is a locally finite open covering of $M$ and has countably many elements. It is not difficult to show by induction that we can obtain $V_\alpha$ by a contraction of $U_\alpha$ such that $\overline{V_\alpha} \subset U_\alpha$ and $\{V_\alpha\}$ is also an open covering for $M$.

By Lemma 3 of §1–3, there exist smooth functions $h_\alpha$, with $0 \leq h_\alpha \leq 1$ on $M$ such that

$$h_\alpha(p) = \begin{cases} 1, & p \in V_\alpha \\ 0, & p \notin U_\alpha \end{cases}$$

(3.13)
§3–3: Integrals of Differential Forms

\{V_\alpha\} forms a covering for \(M\), the point \(p\) must lie in some \(V_\alpha\), i.e., \(h(p) \geq 1\). Let

\[
g_\alpha = \frac{h_\alpha}{h}. \tag{3.14}
\]

Then \(g_\alpha\) is a smooth function on \(M\). It is easy to verify that the family \(\{g_\alpha\}\) satisfies all the conditions of the theorem.

With the above background, we can proceed to define the integration of exterior differential forms on a manifold \(M\). Suppose \(M\) is an \(m\)-dimensional smooth manifold, and \(\varphi\) is an exterior differential \(m\)-form on \(M\) with a compact support. Choose any coordinate covering \(\Sigma = \{W_i\}\) which is consistent with the orientation of \(M\), and suppose that \(\{g_\alpha\}\) is a partition of unity subordinate to \(\Sigma\). Then

\[
\varphi = \left( \sum_\alpha g_\alpha \right) \cdot \varphi = \sum_\alpha (g_\alpha \cdot \varphi). \tag{3.15}
\]

Clearly, \(\text{supp} (g_\alpha \cdot \varphi) \subseteq \text{supp} g_\alpha\) is contained in some coordinate neighborhood \(W_i \in \Sigma\). Therefore we can define

\[
\int_M g_\alpha \cdot \varphi = \int_{W_i} \varphi, \tag{3.16}
\]

where the right hand side is the usual Riemann integral, that is, if \(g_\alpha \cdot \varphi\) with respect to the coordinate system \(u^1, \ldots, u^m\) in \(W_i\) is expressed as

\[
f(u^1, \ldots, u^m) \, du^1 \wedge \cdots \wedge du^m,
\]

then the integral on the right hand side in (3.16) is

\[
\int_{W_i} f(u^1, \ldots, u^m) \, du^1 \cdots du^m. \tag{3.17}
\]

To show that (3.16) is well-defined, we need only show that the right hand side is independent of the choice of \(W_i\). Suppose \(\text{supp} (g_\alpha \cdot \varphi)\) is contained in two coordinate neighborhoods \(W_i\) and \(W_j\), and suppose the local coordinates consistent with the orientation of \(M\) are \(u^k\) and \(v^k\), respectively. Then the Jacobian of the change of coordinates satisfies

\[
J = \frac{\partial(u^1, \ldots, u^m)}{\partial(u^1, \ldots, u^m)} > 0. \tag{3.18}
\]
Suppose \( g_\alpha \cdot \varphi \) is expressed in \( W_i \) and \( W_j \) by
\[
g_\alpha \cdot \varphi = f \, du^1 \wedge \cdots \wedge du^m
\]
respectively. Then
\[
f = f' \cdot J = f' \cdot |J|,
\]
and \( \text{supp } f = \text{supp } f' = \text{supp } (g_\alpha \cdot \varphi) \subset W_i \cap W_j \). By the formula for the change of variables in the Riemann integral, we have
\[
\int_{W_i \cap W_j} f' \, du^1 \cdots du^m = \int_{W_i \cap W_j} f' |J| \, du^1 \cdots du^m = \int_{W_i \cap W_j} f \, du^1 \cdots du^m,
\]
i.e.,
\[
\int_{W_i} g_\alpha \cdot \varphi = \int_{W_j} g_\alpha \cdot \varphi.
\]
Since \( \text{supp } \varphi \) is compact, it only intersects finitely many \( \text{supp } g_\alpha \) by condition 2) of the Partition of Unity Theorem. Therefore the right hand side of (3.15) is a sum of only finitely many terms. Let
\[
\int \varphi = \sum_\alpha \int M g_\alpha \cdot \varphi.
\]
For any given partition of unity \( \{g_\alpha\} \) subordinate to \( \Sigma \), the right hand side of (3.22) is completely determined. Now we show that (3.22) is independent of the choice of the partition of unity \( \{g_\alpha\} \).

Suppose \( \{g'_\beta\} \) is another partition of unity subordinate to \( \Sigma \). Then
\[
\sum_\beta \sum M g'_\beta \cdot \varphi = \sum_\beta \sum_\alpha \sum M g_\alpha \cdot g'_\beta \cdot \varphi = \sum_\beta \sum_\alpha \sum M g'_\beta \cdot g_\alpha \cdot \varphi = \sum_\alpha \sum_\beta \sum M g_\alpha \cdot \varphi.
\]
§3-3: Integrals of Differential Forms

**Definition 3.4.** Suppose $M$ is an $m$-dimensional oriented smooth manifold and $\varphi$ is an exterior differential $m$-form on $M$ with compact support. The numerical value $\int_M \varphi$ defined in (3.22) is called the integral of the exterior differential form $\varphi$ on $M$.

If $\varphi, \varphi_1, \varphi_2$ are exterior differential $m$-forms on $M$ with compact support, then $\varphi_1 + \varphi_2$ has a compact support, as has $c\varphi$, for any real number $c$. By the definition of the integral, it is obvious that

$$\int_M (\varphi_1 + \varphi_2) = \int_M \varphi_1 + \int_M \varphi_2,$$

$$\int_M c\varphi = c\int_M \varphi. \quad (3.23)$$

Therefore the integral $\int_M$ is a linear functional on the set of all exterior differential $m$-forms on $M$ with compact support.

If $\text{supp} \varphi$ happens to be inside a coordinate neighborhood $U$ with local coordinates $u^i$ consistent with the orientation of $M$, then $\varphi$ can be expressed as

$$\varphi = f(u^1, \ldots, u^m)du^1 \wedge \cdots \wedge du^m, \quad (3.24)$$

and $\int_M \varphi$ is precisely the Riemann integral

$$\int_M \varphi = \int_U f du^1 \cdots du^m. \quad (3.25)$$

We see that Definition 3.4 is a generalization of the Riemann integral.

If $\varphi$ is an exterior differential $r$-form, $r < m$, with compact support, then we can define the integral of $\varphi$ on any $r$-dimensional submanifold $N$ of $M$. Suppose

$$h : N \rightarrow M \quad (3.26)$$

is an $r$-dimensional embedding of $N$ into $M$. Then $h^*\varphi$ is an exterior differential $r$-form on the $r$-dimensional smooth manifold $N$, and it has compact support. Therefore the integral $\int_N h^*\varphi$ is well-defined. We define the integral of $\varphi$ on the submanifold $h(N)$ as

$$\int_{h(N)} \varphi = \int_N h^*\varphi. \quad (3.27)$$
§3–4  Stokes’ Formula

The relationship between integrals over a domain and integrals over its boundary is at the heart of the calculus. Let us look at a few examples first.

Example 1. Suppose \( D = [a, b] \) is a closed interval in \( \mathbb{R}^1 \) and \( f \) is a continuously differentiable function on \( D \). Then the Fundamental Theorem of Calculus holds:

\[
\int_D df = f(b) - f(a). \tag{4.1}
\]

We denote the oriented boundary \( \{b\} - \{a\} \) of \( D \) by \( \partial D \). Then the right hand side above can be denoted by \( \int_{\partial D} f \).

Example 2. Suppose \( D \) is a bounded domain in \( \mathbb{R}^2 \) whose orientation is consistent with that of \( \mathbb{R}^2 \). Use \( \partial D \) to denote the oriented boundary of \( D \) with the orientation induced by \( D \), that is, the positive orientation of \( \partial D \) together with the normal vector pointing towards the interior of \( D \) form a coordinate system which is consistent with the orientation of \( \mathbb{R}^2 \) (see Figure 10). Suppose \( P \) and \( Q \) are continuously differentiable functions on \( D \). Then the Green’s Formula holds:

\[
\int_{\partial D} P \ dx + Q \ dy = \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \ dx \ dy. \tag{4.2}
\]
§3-4: Stokes' Formula

If we let $\omega = P \, dx + Q \, dy$, then

$$d\omega = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \wedge dy.$$  

Thus (4.2) can be written as

$$\int_{\partial D} \omega = \int_{D} d\omega. \tag{4.3}$$

Example 3. Suppose $D$ is a bounded domain in $\mathbb{R}^3$ whose orientation is consistent with that of $\mathbb{R}^3$. The outward normal as the positive direction then induces an orientation on the boundary $\partial D$. Suppose $P$, $Q$, and $R$ are continuously differentiable functions on $D$. Then Gauss' formula gives

$$\int_{\partial D} P \, dy \, dz + Q \, dx \, dz + R \, dx \, dy = \int_{D} \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) \, dx \, dy \, dz, \tag{4.4}$$

or,

$$\int_{\partial D} \varphi = \int_{D} d\varphi, \tag{4.5}$$

where $\varphi = P \, dy \wedge dz + Q \, dx \wedge dz + R \, dx \wedge dy$.

Example 4. Suppose $\Sigma$ is an oriented surface in $\mathbb{R}^3$ whose boundary $\partial \Sigma$ is an oriented closed curve. The positive orientation of $\partial \Sigma$ along with any positive normal vector to $\Sigma$ satisfy the right-hand rule (assuming $\mathbb{R}^3$ to be oriented by a right-handed system). Suppose $P$, $Q$, and $R$ are continuously differentiable functions on a domain containing $\Sigma$. Then the Stokes' formula holds:

$$\int_{\partial \Sigma} P \, dx + Q \, dy + R \, dz = \int_{\Sigma} \left\{ \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \, dy \, dz \right\}$$

$$+ \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \, dz \, dx + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy \right\}.$$  

If we denote

$\omega = P \, dx + Q \, dy + R \, dz$,

then the above formula can be rewritten as

$$\int_{\partial \Sigma} \omega = \int_{\Sigma} d\omega. \tag{4.7}$$
Chapter 3: Exterior Differential Calculus

We see that the above four formulas assume a unified form in the notation of exterior differential forms. The Stokes formula considered in this section is the generalization of the above formulas for manifolds. First we need to explain some essential concepts.

Definition 4.1. Suppose \( M \) is an \( m \)-dimensional smooth manifold. A region \( D \) with boundary is a subset of the manifold \( M \) with two kinds of points:

1) Interior points, each of which has a neighborhood in \( M \) contained in \( D \).

2) Boundary points \( p \), for each of which there exists a coordinate chart \((U; u^i)\) such that \( u^i(p) = 0 \) and

\[
U \cap D = \{ q \in U | u^m(q) \geq 0 \}. \tag{4.8}
\]

A coordinate system \( u^i \) with the above property is called an adapted coordinate system for the boundary point \( p \).

The set of all the boundary points of \( D \) is called the boundary of \( D \), denoted by \( B \).

Theorem 4.1. The boundary \( B \) of a region \( D \) with boundary is a regular imbedded closed submanifold. If \( M \) is orientable, then \( B \) is also orientable.

Proof. The boundary \( B \) of the region \( D \) is obviously a closed subset of \( M \). Suppose \((U; u^i)\) is an adapted coordinate neighborhood. Then

\[
U \cap B = \{ q \in U | u^m(q) = 0 \}. \tag{4.9}
\]

By Definition 3.2 in Chapter 1, \( B \) is a regular imbedded closed submanifold of \( M \).

Suppose \( M \) is an oriented manifold. Choose an adapted coordinate neighborhood \((U; u^i)\) which is consistent with the orientation of \( M \) at an arbitrary point \( p \in B \). Then \((u^1, \ldots, u^{m-1})\) is a local coordinate system of \( B \) at the point \( p \). Let

\[
(-1)^m du^1 \wedge \cdots \wedge du^{m-1} \in \Omega^m(\mathbb{R}^{m-1}) \tag{4.10}
\]

specify the orientation of the boundary \( B \) in the coordinate neighborhood \( U \cap B \) of the point \( p \). We will prove that the orientations given in this way to the coordinate neighborhoods are consistent. Suppose \((V; v^i)\) is another coordinate neighborhood of a boundary point \( p \) consistent with the orientation of \( M \). Then

\[
\frac{\partial (v^1, \ldots, v^m)}{\partial (u^1, \ldots, u^{m-1})} > 0. \tag{4.11}
\]
This shows that \((-1)^m du^1 \wedge \cdots \wedge du^{m-1}\) and \((-1)^m dv^1 \wedge \cdots \wedge dv^{m-1}\) give consistent orientations in \(U \cap V \cap B\). Hence \(B\) is orientable. \(\square\)

The orientation of \(B\) given in (4.10) is called the induced orientation on the boundary \(B\) by an oriented manifold \(M\). If \(D\) has the same orientation as \(M\) we denote the boundary \(B\) with the induced orientation by \(\partial D\). It is easy to verify that the orientations of \(\partial D\) and \(\partial \Sigma\) in the preceding four examples are induced in this way.

**Theorem 4.2 (Stokes' Formula):** Suppose \(D\) is a region with boundary in an \(m\)-dimensional oriented manifold \(M\), and \(\omega\) is an exterior differential \((m-1)\)-form on \(M\) with compact support. Then

\[
\int_D d\omega = \int_{\partial D} \omega. \quad (4.13)
\]

If \(\partial D = \emptyset\), then the integral on the right hand side is zero.

**Proof.** Suppose \(\{U_i\}\) is a coordinate covering consistent with the orientation of \(M\), and \(\{g_\alpha\}\) is a subordinate partition of unity. Then

\[
\omega = \sum_\alpha g_\alpha \cdot \omega. \quad (4.14)
\]

Since \(\text{supp } \omega\) is compact, the right hand side of the above formula is a sum of finitely many terms. Therefore

\[
\int_D d\omega = \sum_\alpha \int_D d(g_\alpha \cdot \omega),
\]

\[
\int_{\partial D} \omega = \sum_\alpha \int_{\partial D} g_\alpha \cdot \omega. \quad (4.15)
\]

This implies that we only need to prove

\[
\int_D d(g_\alpha \cdot \omega) = \int_{\partial D} g_\alpha \cdot \omega. \quad (4.16)
\]
for each \( \alpha \). We may assume that \( \text{supp} \, \omega \) is contained in a coordinate neighborhood \((U; u^i)\) consistent with the orientation of \( M \). Suppose \( \omega \) can be expressed as

\[
\omega = \sum_{j=1}^{m} (-1)^{j-1} a_j du^1 \wedge \cdots \wedge \widehat{du^j} \wedge \cdots \wedge du^m,
\]

(4.17)

where the \( a_j \) are smooth functions on \( U \). Then

\[
d\omega = \left( \sum_{j=1}^{m} \frac{\partial a_j}{\partial u^j} \right) du^1 \wedge \cdots \wedge du^m.
\]

(4.18)

There are two cases to consider.

1) If \( U \cap \partial D = \emptyset \), the right hand side of (4.13) is zero. Then either \( U \) is contained in \( M - D \) or in the interior of \( D \). In the former case, the left hand side of (4.13) is obviously zero. In the latter, we have

\[
\int_D d\omega = \sum_{j=1}^{m} \int_U \frac{\partial a_j}{\partial u^j} du^1 \cdots du^m.
\]

(4.19)

Consider a cube \( C = \{ u \in \mathbb{R}^m | |u^i| \leq K, 1 \leq i \leq m \} \) such that \( U \) is contained in \( C \). Extend the functions \( a_j \) to \( C \) by letting them be zero outside \( U \). Obviously the \( a_j \) are continuously differentiable in \( C \). Hence

\[
\int_U \frac{\partial a_j}{\partial u^j} du^1 \cdots du^m = \int_C \frac{\partial a_j}{\partial u^j} du^1 \cdots du^m
\]

\[
= \int_{|u^i| \leq K \atop i \neq j} \left( \int_{-K}^{+K} \frac{\partial a_j}{\partial u^j} du^j \right) du^1 \cdots du^{j-1} du^{j+1} \cdots du^m
\]

\[
= 0.
\]

(4.20)

The last integral above vanishes since

\[
\int_{-K}^{+K} \frac{\partial a_j}{\partial u^j} du^j = a_j(u^1, \ldots, u^{j-1}, K, u^{j+1}, \ldots, u^m)
\]

\[
- a_j(u^1, \ldots, u^{j-1}, -K, u^{j+1}, \ldots, u^m)
\]

(4.21)

\[
= 0.
\]
2) If \( U \cap \partial D \neq \emptyset \), we may assume that \( U \) is an adapted coordinate neighborhood consistent with the orientation of \( M \). Then we have

\[
U \cap D = \{ q \in U | u^m(q) \geq 0 \}, \quad \text{and} \quad (4.22)
\]

\[
U \cap \partial D = \{ q \in U | u^m(q) = 0 \}. \quad (4.23)
\]

Choose a cube \( C = \{ u \in \mathbb{R}^m | |u^i| \leq K, 1 \leq i \leq m - 1; 0 \leq u^m \leq K \} \).

When \( K \) is sufficiently large, \( U \cap D \) lies in the union of the interior of \( C \) and the boundary \( u^m = 0 \). Extend \( a_j \) as in 1). Then the right hand side of (4.13) becomes

\[
\int_{\partial D} \omega = \int_{U \cap \partial D} \omega
\]

\[
= \sum_{j=1}^{m-1} (-1)^{j-1} \int_{U \cap \partial D} a_j \, du^1 \wedge \cdots \wedge du^{j-1} \wedge du^{j+1} \wedge \cdots \wedge du^m
\]

\[
= (-1)^{m-1} \int_{U \cap \partial D} a_m \, du^1 \wedge \cdots \wedge du^{m-1} \int_{\gamma \cap \partial D} \left[ \sum a_m (u^1, \ldots, u^{m-1}, 0) du^1 \cdots du^{m-1} \right]
\]

\[
= \int_{|u| \leq K, 1 \leq i \leq m-1} \left( (-1)^m \sum a_j \right) du^1 \cdots du^m
\]

(4.24)

where the third equality holds since \( du^m = 0 \) on \( U \cap \partial D \), while in the last equality we have used the induced orientation on \( \partial D \) by \( M \).

The left-hand side of (4.13) is

\[
\int_D \omega = \int_{D \cap U} \omega
\]

\[
= \sum_{j=1}^{m} \int_{D \cap U} \frac{\partial a_j}{\partial u^j} du^1 \wedge \cdots \wedge du^m.
\]

(4.25)
But for \( 1 \leq j \leq m - 1 \),
\[
\int_{\partial U} \frac{\partial a_j}{\partial u^j} du^1 \wedge \ldots \wedge du^m
= \int_{|u^1| \leq K, i \neq j, m} \left( \int_{-K}^{+K} \frac{\partial a_j}{\partial u^j} du^j \right) du^1 \ldots du^{j-1} du^{j+1} \ldots du^m
\]
\[
= 0.
\]
(4.26)

Thus there is only one nonzero term in (4.25):
\[
\int_{\partial D \cap U} \frac{\partial a_m}{\partial u^m} du^1 \wedge \ldots \wedge du^m
= \int_{|u^1| \leq K, i \neq m} (a_m(u^1, \ldots, u^{m-1}, K) - a_m(u^1, \ldots, u^{m-1}, 0)) du^1 \ldots du^{m-1}
\]
\[
= - \int_{|u^1| \leq K, i \neq m} a_m(u^1, \ldots, u^{m-1}, 0) du^1 \ldots du^{m-1}.
\]
(4.27)

Hence (4.13) is true, and the theorem is proved.

Remark. In most applications, the closed region \( D \) is compact. In such cases we need not assume that the exterior differential \((m - 1)\)-form has a compact support for the Stokes' formula to be valid.

The Stokes formula plays an important role in physics, mechanics, partial differential equations, and differential geometry. Integration is additive with respect to the domain of integration, hence we can further define the integral of exterior differential forms on singular chains. Viewing the integral as a pairing between an exterior differential form and a domain of integration, every exterior differential form is equivalent to a singular cochain on the manifold \( M \), and the Stokes' formula describes the duality between the boundary operator \( \partial \) and the coboundary operator \( d \). If we denote
\[
(\partial D, \omega) = \int_D \omega,
\]
\[
(D, d\omega) = \int_D d\omega,
\]
(4.28)

---

*See, for example Singer and Thorpe 1976.*