The Dictionary Between Nevanlinna Theory and Diophantine approximation

0.1 Valuations on a number field

Definition 1 *Let F be a field. By an* **absolute value** *on F, we mean a real-valued function $|\ |$ on F satisfying the following three conditions:*
(i) $|a| \geq 0$, *and* $|a| = 0$ *if and only if* $a = 0$.
(ii) $|ab| = |a||b|$.
(iii) $|a + b| \leq |a| + |b|$.

Two absolute values $|\ |_1$ and $|\ |_2$ are called **equivalent** if there is a positive constant λ such that $|\ |_1 = |\ |_2^{\lambda}$. Over the field of rational numbers $\mathbf{Q}$ we have the following absolute values: **the standard Archimedean absolute value** $|\ |$ (we also denote it by $|\ |_\infty$), which is defined by $|x| = x$ if $x \geq 0$, and $|x| = -x$ if $x < 0$; **p-adic absolute value** $|\ |_p$, for each prime number p, defined by $|x|_p = p^{-r}$, if $x = p^r a/b$, for some integer r, where a and b are integers relatively prime to p. For $x = 0$, $|x|_p = 0$. The p-adic absolute value $|\ |_p$ satisfies (i) and (ii), and a property stronger than (iii) in Definition 1, namely

$$(iii)' \qquad |a + b|_p \leq \max\{|a|_p, |b|_p\}.$$

An absolute value that satisfies $(iii)'$ is called a **non-Archimedean absolute value**. Every nonzero rational number has a factorization into prime factors. So for every $x \in \mathbf{Q}$ with $x \neq 0$, we have

$$|x|_\infty \cdot \prod_p |x|_p = 1, \tag{1}$$

where in the product, p runs for all prime numbers. (1) is called the **product formula**.

Theorem 2 (A. Ostrowski) *Any absolute value on $\mathbf{Q}$ is equivalent to one of the following: a p-adic absolute value for some prime number p, the standard Archimedean absolute value $|\ |_\infty$, or the trivial absolute value $|\ |_0$ defined by $|x|_0 = 1$ for all $x \neq 0$.*

To clearly see how Roth's theorem connects to Nevanlinna theory, we have to consider the fields more general than $\mathbf{Q}$, namely the number fields. Let us first consider the extension of an absolute value to $\mathbf{Q}(\alpha)$ where α is an algebraic number. We know that an algebraic number is usually viewed as a complex roots of its minimal polynomial. Then $|\alpha|$ is just the modulus of this complex number, and extends $| \ |_\infty$ to an absolute value of $\mathbf{Q}(\alpha)$. To extend a p-adic absolute value is less easy. But if one is willing to accept the p-adic closure $\mathbf{Q}_p$ of $\mathbf{Q}$ and the algebraic closure $\mathbf{C}_p$ of $\mathbf{Q}_p$, with the corresponding extension of $| \ |_p$ to $\mathbf{C}_p$, this becomes just as easy as for $| \ |_\infty$. Namely, every embedding $\sigma : \mathbf{Q}(\alpha) \to \mathbf{C}_p$ gives an extension of $| \ |_p$ defined by $|\beta|_p = |\sigma(\beta)|_p$, for $\beta \in \mathbf{Q}(\alpha)$. More precisely, we present, in the following, the theory of the extension of absolute values to a number field k. A number field k is a finite extension of the rationals $\mathbf{Q}$. Absolute values on $\mathbf{Q}$ extend to absolute values on k. The absolute values on k are divided into Archimedeans and non-Archimedeans. The Archimedean absolute values arise in the following ways: Let $n = [k : \mathbf{Q}]$. It is a standard fact from the field theory that k admits exactly n distinct embeddings $\sigma : k \hookrightarrow \mathbf{C}$. Each such embedding is used to define an absolute value on k according to the rule

$$|x|_\sigma = |\sigma(x)|_\infty$$

where $| \ |_\infty$ is the usual absolute value on $\mathbf{C}$. Recall that the embeddings $\sigma : k \hookrightarrow \mathbf{C}$ come in two flavors, the real embeddings (i.e., $\sigma(k) \subset \mathbf{R}$) and complex embeddings (i.e. $\sigma(k) \not\subset \mathbf{R}$). The complex embeddings come in pairs that differ by complex conjugation. The usual notation is that there are r_1 real embeddings and $2r_2$ pairs of complex embeddings, so $n = r_1 + 2r_2$. The normalized almost absolute value corresponding to σ is then defined by

$$\|x\|_\sigma = |x|_\sigma, \tag{2}$$

if σ is a real embedding, and

$$\|x\|_\sigma = |x|_\sigma^2, \tag{3}$$

if σ is a complex embedding. We note that the normalized almost-absolute values arising from the complex embedding do not satisfy the triangle inequality. This is why they are called almost-absolute values.

The non-Archimedean absolute values on k arise in much the same way as they do on $\mathbf{Q}$. However, one may not be able to uniquely factor elements of k into primes. A key idea in number theory is to look at prime ideals instead. To be more precise, let $\mathbf{R}_k$ be the ring of algebraic integers of k. Recall that $x \in k$ is called an **algebraic integer** if x is a root of a monic polynomial with coefficients in $\mathbf{Z}$. Note that, although $\mathbf{R}_k$ is not a principle ideal domain, for every $x \in \mathbf{R}_k$, the principal ideal (x) in $\mathbf{R}_k$ generated by x does factor uniquely into a product of prime ideals. For every prime ideal $\mathcal{P}$ of $\mathbf{R}_k$, we denote by $\text{ord}_{\mathcal{P}}x$ the number of times the prime ideal $\mathcal{P}$ appears in this ideal factorization. Every prime ideal $\mathcal{P}$ lies above some prime p in $\mathbf{Q}$. For every element $x \in \mathbf{R}_k$, we define

$$|x|_{\mathcal{P}} = p^{-\text{ord}_{\mathcal{P}}\text{x}/\text{ord}_{\mathcal{P}}\text{p}}.$$

Of course, we always understand that $\text{ord}_{\mathcal{P}}0 = \infty$. The absolute value $|\ |_{\mathcal{P}}$ extends to k by writing any $x \in k$ as the quotient of two elements in $\mathbf{R}_k$. Note that the $\text{ord}_{\mathcal{P}}\text{p}$ is needed to ensure that $|p|_{\mathcal{P}} = p^{-1}$. To get the normalized non-Archimedean absolute values, let $\mathbf{Q}_p$ be the completion of $\mathbf{Q}$ with respect to the p-adic absolute value $|\ |p$ on $\mathbf{Q}$ and $k_{\mathcal{P}}$the completion of k with respect to $|\ |_{\mathcal{P}}$. For every element $x \in \mathbf{R}_k$, we define the normalized norm

$$\|x\|_{\mathcal{P}} = |x|_{\mathcal{P}}^{[k_{\mathcal{P}}:\mathbf{Q}_p]}. \tag{4}$$

The absolute value $\|\ \|_{\mathcal{P}}$ extends to k by writing any $x \in k$ as the quotient of two elements in $\mathbf{R}_k$. Note the definition in (4) can also be written as

$$\|x\|_{\mathcal{P}} = \left(N_{k/\mathbf{Q}}\mathcal{P}\right)^{-\text{ord}_{\mathcal{P}}\text{x}},$$

where $N_{k/\mathbf{Q}}\mathcal{P}$ is the norm of the ideal $\mathcal{P}$.

Theorem 2 is then extended to the following theorem.

Theorem 3 (A. Ostrowski) *Let k be a number field. Any almost-absolute value on k is equivalent to one of the following: the Archimedean absolute values which come from the real embeddings $\sigma : k \to \mathbf{R}$ defined by (2); the Archimedean almost-absolute values which come from the complex embeddings $\sigma : k \to \mathbf{C}$ defined by (2); and the non-Archimedean absolute value $\|\ \|_{\mathcal{P}}$ for some prime number $p \in \mathbf{Q}$, defined by (2).*

We refer to the real embeddings $\sigma : k \to \mathbf{R}$, the complex conjugate pairs $\{\sigma, \bar{\sigma}\}$ of the complex embeddings $\sigma : k \to \mathbf{C}$, and the nonzero prime ideals

$\mathcal{P}$ in the ring $\mathbf{R}_k$ as **real places**, **complex places** and **non-Archimedean places**. We denote by M_k the canonical set of all the non-equivalent places. The set of non-equivalent Archimedean places of k is denoted by M_k^∞, the set of non-equivalent non-Archimedean places of k is denoted by M_k^0. For every place $v \in M_k$, v has **almost-absolute values** $\|\cdot\|_v$ defined by

$$\|x\|_v = \begin{cases} |\sigma(x)| & \text{if } v \text{ is real, corresponding to } \sigma : \mathrm{k} \to \mathbf{R} \\ |\sigma(x)|^2 & \text{if } v \text{ is complex, corresponding to } \sigma, \bar{\sigma} : \mathrm{k} \to \mathbf{C} \\ \left(N_{k/\mathbf{Q}}\mathcal{P}\right)^{-\mathrm{ord}_{\mathcal{P}}\mathrm{x}} & \text{if } v \text{ is non-Arch., corresponding to } \mathcal{P} \subset \mathbf{R}_\mathrm{k} \end{cases} \tag{5}$$

for $x \neq 0 \in k$. We also define $\|0\|_v = 0$. As we noted, these are not necessarily genuine absolute values. However, instead of having the triangle inequality, we have a value such that if $a_1, \ldots, a_n \in k$, then

$$\left\|\sum_{i=1}^{n} a_i\right\|_v \leq n^{N_v} \max_{1 \leq i \leq n} \|a_i\|_v, \tag{6}$$

where

$$N_v = \begin{cases} 1 & \text{if } v \text{ is real} \\ 2 & \text{if } v \text{ is complex} \\ 0 & \text{if } v \text{ is non-Archimedean.} \end{cases}$$

If L is a finite extension of k, $v \in M_k$, and $x \in k$, then

$$\prod_{w \in M_L, w|v} \|x\|_w = \|x\|_v^{[L:k]}. \tag{7}$$

Artin-Whaples extended the product formula on $\mathbf{Q}$ to the number fields.

Theorem 4 (Product Formula) *Let k be a number field. Let M_k be the canonical set of non-equivalent places on k. Then, for every $x \in k$ with $x \neq 0$,*

$$\prod_{v \in M_k} \|x\|_v = 1. \tag{8}$$

0.2 Roth's Theorem

Roth's theorem was extended by Mahler to number field k as follows:

Theorem 5 (Roth) *Given $\epsilon > 0$, a finite set of places S of k containing M_k^∞, and $\alpha_v \in \overline{\mathbf{Q}}$ for each $v \in S$. Then for all, except for finitely many, $x \in k$,*

$$\frac{1}{[k:\mathbf{Q}]}\sum_{v\in S} -\log\min(\|x-\alpha_v\|_v, 1) \leq (2+\epsilon)h(x), \tag{9}$$

where $h(x)$ is the absolute logarithmic height defined by

$$h(x) = \frac{1}{[k:\mathbf{Q}]}\sum_{v\in M_k} \log^+ \|x\|_v. \tag{10}$$

Fix a finite set S containing M_k^∞, we define, for $a, x \in k$,

$$m(x,a) = \frac{1}{[k:\mathbf{Q}]}\sum_{v\in S} \log^+ \frac{1}{\|x-a\|_v}, \tag{11}$$

$$N(x,a) = \frac{1}{[k:\mathbf{Q}]}\sum_{v\notin S} \log^+ \frac{1}{\|x-a\|_v}. \tag{12}$$

Then the product formula (Theorem 4) reads

Theorem 6 *For all $x \in k^*$, $a \in k$*

$$m(x,a) + N(x,a) = h(x) + O(1).$$

Theorem B1.2.5 can be restated as

Theorem 7 (Roth) *Given $\epsilon > 0$, a finite set $S \subset M_k$ containing M_k^∞, and distinct points $a_1, \ldots, a_q \in k$. Then the inequality*

$$\sum_{j=1}^{q} m(x,a_j) \leq (2+\epsilon)h(x)$$

holds for all, except for finitely many, $x \in k$.

Lang made the following conjecture with a more precise error term.

Conjecture 8 (Lang) *Given $\epsilon > 0$, a finite set $S \subset M_k$ containing M_k^∞, and distinct points $a_1, \ldots, a_q \in k$, the inequality*

$$(q-2)h(x) \leq \sum_{j=1}^{q} N^{(1)}(x, a_j) + (1+\epsilon)\log h(x)$$

holds for all, except for finitely many, $x \in k$.

Roth's theorem implies the following analogy of Picard's Theorem.

Theorem 9 *Let k be a number field, and let $a_1, \ldots, a_q$ be distinct numbers in $k \cup \{\infty\}$. If $q \geq 3$, then there are only finitely many elements $x \in k$ such that $1/(x - a_j)$ (or x itself if $a_j = \infty$) is an algebraic integer for all $1 \leq j \leq q$.*

To further explore the analogy, we introduce more notation. Recall the Nevanlinna counting function for a meromorphic function f is defined by

$$N_f(r, a) = \sum_{z \in \mathbf{D}(r), z \neq 0} \mathrm{ord}_z^+(f - a) \log \frac{r}{|z|} + \mathrm{ord}_0^+(f - a) \log r.$$

On the other hand, take $S = M_k^\infty$, then the number theoretic counting function $N(x, a)$ defined by (12) can be rewritten as

$$\begin{aligned} N(x,a) &= \frac{1}{[k:\mathbf{Q}]} \sum_{v \notin M_k^\infty} \log^+ \frac{1}{\|x - a\|_v} \\ &= \frac{1}{[k:\mathbf{Q}]} \sum_{\mathcal{P} \subset \mathbf{R}_k} \mathrm{ord}_{\mathcal{P}}^+(x - a) \log(N_{k/\mathbf{Q}}\mathcal{P}) \qquad (13) \end{aligned}$$

where $\mathrm{ord}_{\mathcal{P}}^+ \mathrm{x} = \max\{0, \mathrm{ord}_{\mathcal{P}} \mathrm{x}\}$. So $N_f(r, a)$ and $N(x, a)$ can be compared by replacing $\log(r/|z|)$ in the definition of $N_f(r, a)$ with $\log(N_{k/\mathbf{Q}}\mathcal{P})$ in the definition of $N(x, a)$. From this point of view, Paul Vojta has compiled a dictionary to translate the terms in Nevanlinna theory to the terms in Diophantine approximation. It is provided on p. 32.

Vojta's Dictionary

Nevanlinna Theory	**Diophantine Approximation**
non-constant meromorphic function f	infinite $\{x\}$ in a number field k
A radius r	An element of k
A finite measure set E of radii	A finite subset of $\{x\}$
An angle θ	An embedding $\sigma : k \to \mathbf{C}$
$\lvert f(re^{i\theta})\rvert$	$\lvert x\rvert_\sigma$
$(\mathrm{ord}_z f)\log\frac{r}{\lvert z\rvert}$	$(\mathrm{ord}_{\mathcal{P}} x)\log(N_{k/\mathbf{Q}}\mathcal{P})$
Proximity function	Proximity function
$m_f(r,a) = \int_0^{2\pi}\log^+\left\lvert\frac{1}{f(re^{i\theta})-a}\right\rvert\frac{d\theta}{2\pi}$	$m(x,a) = \sum_{\sigma:k\to\mathbf{C}}\log^+\left\lVert\frac{1}{x-a}\right\rVert_\sigma$
Counting function:	Counting function:
$N_f(r,a) = \mathrm{ord}_0^+(\mathrm{f}-\mathrm{a})\log\mathrm{r}$ $+\sum_{0<\lvert z\rvert<r}\mathrm{ord}_{\mathrm{z}}^+(\mathrm{f}-\mathrm{a})\log\frac{\mathrm{r}}{\lvert\mathrm{z}\rvert}$	$N(x,a) =$ $\frac{1}{[k:\mathbf{Q}]}\sum_{\mathcal{P}\subset\mathbf{R}_k}\mathrm{ord}_{\mathcal{P}}^+(\mathrm{x}-\mathrm{a})\log(\mathrm{N}_{\mathrm{k}/\mathbf{Q}}\mathcal{P})$
Characteristic function	Logarithmic height
$T_f(r) = \int_0^{2\pi}\log^+\lvert f(re^{i\theta})\rvert\frac{d\theta}{2\pi}$ $+N_f(r,\infty)$	$h(x) = \frac{1}{[k:\mathbf{Q}]}\sum_{\sigma:k\to\mathbf{C}}\log^+\lVert x\rVert_\sigma$ $+N(x,\infty)$
Jensen's formula:	Atin-Whaples Product Formula:
$\int_0^{2\pi}\log\lvert f(re^{i\theta})\rvert\frac{d\theta}{2\pi}$ $= N_f(r,0) - N_f(r,\infty) + O(1)$	$\sum_{\sigma:k\to\mathbf{C}}\log\lVert x\rVert_\sigma$ $= N(x,0) - N(x,\infty)$
First Main Theorem:	Height Property:
$m_f(r,a) + N_f(r,a) = T_f(r) + O(1)$	$m(x,a) + N(x,a) = h(x) + O(1)$
Weaker Second Main Theorem:	Roth's Theorem:
$(q-2)T_f(r) - \sum_{j=1}^{q}N_f(r,a_j) .\le. \epsilon T_f(r)$	$(q-2)h(x) - \sum_{j=1}^{q}N(x,a_j) .\le. \epsilon h(x)$
Second Main Theorem:	Lang's conjecture:
$(q-2)T_f(r) - \sum_{j=1}^{q}N_f^{(1)}(r,a_j)$ $.\le. (1+\epsilon)\log T_f(r)$	$(q-2)h(x) - \sum_{j=1}^{q}N^{(1)}(x,a_j)$ $.\le. (1+\epsilon)\log h(x)$

Note that, in above, we use the notation $.\le.$ to denote that the inequality holds for all r except a set $E \subset (0,+\infty)$ with finite Lebesgue measure in Nevanlinna theory and the inequality holds for all, except for finitely many, $x \in k$ in Diophantine approximation.