0.1 Valuations on a number field

**Definition 1** Let $F$ be a field. By an absolute value on $F$, we mean a real-valued function $|\cdot|$ on $F$ satisfying the following three conditions:

(i) $|a| \geq 0$, and $|a| = 0$ if and only if $a = 0$.
(ii) $|ab| = |a||b|$.
(iii) $|a + b| \leq |a| + |b|$.

Two absolute values $|\cdot|_1$ and $|\cdot|_2$ are called equivalent if there is a positive constant $\lambda$ such that $|\cdot|_1 = |\cdot|_2^\lambda$. Over the field of rational numbers $\mathbb{Q}$ we have the following absolute values: the standard Archimedean absolute value $|\cdot|_\infty$ (we also denote it by $|\cdot|_\infty$), which is defined by $|x| = x$ if $x \geq 0$, and $|x| = -x$ if $x < 0$; p-adic absolute value $|\cdot|_p$, for each prime number $p$, defined by $|x|_p = p^{-r}$, if $x = p^ra/b$, for some integer $r$, where $a$ and $b$ are integers relatively prime to $p$. For $x = 0$, $|x|_p = 0$. The $p$-adic absolute value $|\cdot|_p$ satisfies (i) and (ii), and a property stronger than (iii) in Definition 1, namely

$$(iii)' \quad |a + b|_p \leq \max\{|a|_p, |b|_p\}.$$

An absolute value that satisfies $(iii)'$ is called a non-Archimedean absolute value. Every nonzero rational number has a factorization into prime factors. So for every $x \in \mathbb{Q}$ with $x \neq 0$, we have

$$|x|_\infty \cdot \prod_p |x|_p = 1,$$

(1)

where in the product, $p$ runs for all prime numbers. (1) is called the product formula.

**Theorem 2 (A. Ostrowski)** Any absolute value on $\mathbb{Q}$ is equivalent to one of the following: a $p$-adic absolute value for some prime number $p$, the standard Archimedean absolute value $|\cdot|_\infty$, or the trivial absolute value $|\cdot|_0$ defined by $|x|_0 = 1$ for all $x \neq 0$. 

1
To clearly see how Roth’s theorem connects to Nevanlinna theory, we have to consider the fields more general than \( \mathbb{Q} \), namely the number fields. Let us first consider the extension of an absolute value to \( \mathbb{Q}(\alpha) \) where \( \alpha \) is an algebraic number. We know that an algebraic number is usually viewed as a complex root of its minimal polynomial. Then \( |\alpha| \) is just the modulus of this complex number, and extends \( |\cdot|_\infty \) to an absolute value of \( \mathbb{Q}(\alpha) \). To extend a \( p \)-adic absolute value is less easy. But if one is willing to accept the \( p \)-adic closure \( \mathbb{Q}_p \) of \( \mathbb{Q} \) and the algebraic closure \( \mathbb{C}_p \) of \( \mathbb{Q}_p \), with the corresponding extension of \( |\cdot|_p \) to \( \mathbb{C}_p \), this becomes just as easy as for \( |\cdot|_\infty \). Namely, every embedding \( \sigma : \mathbb{Q}(\alpha) \to \mathbb{C}_p \) gives an extension of \( |\cdot|_p \) defined by \( |\beta|_p = |\sigma(\beta)|_p \), for \( \beta \in \mathbb{Q}(\alpha) \). More precisely, we present, in the following, the theory of the extension of absolute values to a number field \( k \). A number field \( k \) is a finite extension of the rationals \( \mathbb{Q} \). Absolute values on \( \mathbb{Q} \) extend to absolute values on \( k \). The absolute values on \( k \) are divided into Archimedean and non-Archimedean. The Archimedean absolute values arise in the following ways: Let \( n = [k : \mathbb{Q}] \). It is a standard fact from the field theory that \( k \) admits exactly \( n \) distinct embeddings \( \sigma : k \hookrightarrow \mathbb{C} \). Each such embedding is used to define an absolute value on \( k \) according to the rule

\[
|x|_\sigma = |\sigma(x)|_\infty
\]

where \( |\cdot|_\infty \) is the usual absolute value on \( \mathbb{C} \). Recall that the embeddings \( \sigma : k \hookrightarrow \mathbb{C} \) come in two flavors, the real embeddings (i.e., \( \sigma(k) \subset \mathbb{R} \)) and complex embeddings (i.e. \( \sigma(k) \not\subset \mathbb{R} \)). The complex embeddings come in pairs that differ by complex conjugation. The usual notation is that there are \( r_1 \) real embeddings and \( 2r_2 \) pairs of complex embeddings, so \( n = r_1 + 2r_2 \). The normalized almost absolute value corresponding to \( \sigma \) is then defined by

\[
\|x\|_\sigma = |x|_\sigma, \quad (2)
\]

if \( \sigma \) is a real embedding, and

\[
\|x\|_\sigma = |x|_\sigma^2, \quad (3)
\]

if \( \sigma \) is a complex embedding. We note that the normalized almost-absolute values arising from the complex embedding do not satisfy the triangle inequality. This is why they are called almost-absolute values.

The non-Archimedean absolute values on \( k \) arise in much the same way as they do on \( \mathbb{Q} \). However, one may not be able to uniquely factor elements
of \( k \) into primes. A key idea in number theory is to look at prime ideals instead. To be more precise, let \( \mathbf{R}_k \) be the ring of algebraic integers of \( k \). Recall that \( x \in k \) is called an **algebraic integer** if \( x \) is a root of a monic polynomial with coefficients in \( \mathbb{Z} \). Note that, although \( \mathbf{R}_k \) is not a principle ideal domain, for every \( x \in \mathbf{R}_k \), the principal ideal \((x) \) in \( \mathbf{R}_k \) generated by \( x \) does factor uniquely into a product of prime ideals. For every prime ideal \( \mathcal{P} \) of \( \mathbf{R}_k \), we denote by \( \text{ord}_{\mathcal{P}} x \) the number of times the prime ideal \( \mathcal{P} \) appears in this ideal factorization. Every prime ideal \( \mathcal{P} \) lies above some prime \( p \) in \( \mathbb{Q} \). For every element \( x \in \mathbf{R}_k \), we define

\[
|x|_\mathcal{P} = p^{-\text{ord}_{\mathcal{P}} x / \text{ord}_{\mathcal{P}} p}.
\]

Of course, we always understand that \( \text{ord}_{\mathcal{P}} 0 = \infty \). The absolute value \( |\cdot|_\mathcal{P} \) extends to \( k \) by writing any \( x \in k \) as the quotient of two elements in \( \mathbf{R}_k \). Note that the \( \text{ord}_{\mathcal{P}} p \) is needed to ensure that \( |p|_\mathcal{P} = p^{-1} \). To get the normalized non-Archimedean absolute values, let \( \mathbb{Q}_p \) be the completion of \( \mathbb{Q} \) with respect to the \( p \)-adic absolute value \( |\cdot|_p \) on \( \mathbb{Q} \) and \( k_p \) the completion of \( k \) with respect to \( |\cdot|_p \). For every element \( x \in \mathbf{R}_k \), we define the normalized norm

\[
\|x\|_\mathcal{P} = |x|_\mathcal{P}^{[k_p:Q_p]}.
\]

The absolute value \( \|\cdot\|_\mathcal{P} \) extends to \( k \) by writing any \( x \in k \) as the quotient of two elements in \( \mathbf{R}_k \). Note the definition in (4) can also be written as

\[
\|x\|_\mathcal{P} = \left( N_{k/Q} \mathcal{P} \right)^{-\text{ord}_{\mathcal{P}} x},
\]

where \( N_{k/Q} \mathcal{P} \) is the norm of the ideal \( \mathcal{P} \).

Theorem 2 is then extended to the following theorem.

**Theorem 3 (A. Ostrowski)** Let \( k \) be a number field. Any almost-absolute value on \( k \) is equivalent to one of the following: the Archimedean absolute values which come from the real embeddings \( \sigma : k \to \mathbb{R} \) defined by (2); the Archimedean almost-absolute values which come from the complex embeddings \( \sigma : k \to \mathbb{C} \) defined by (2); and the non-Archimedean absolute value \( \|\cdot\|_p \) for some prime number \( p \in \mathbb{Q} \), defined by (2).

We refer to the real embeddings \( \sigma : k \to \mathbb{R} \), the complex conjugate pairs \( \{\sigma, \overline{\sigma}\} \) of the complex embeddings \( \sigma : k \to \mathbb{C} \), and the nonzero prime ideals
\[ P \] in the ring \( \mathbb{R}_k \) as real places, complex places and non-Archimedean places. We denote by \( M_k \) the canonical set of all the non-equivalent places. The set of non-equivalent Archimedean places of \( k \) is denoted by \( M_{k}^\infty \), the set of non-equivalent non-Archimedean places of \( k \) is denoted by \( M^0_k \). For every place \( \nu \in M_k \), \( \nu \) has almost-absolute values \( \| \cdot \|_\nu \) defined by

\[
\| x \|_\nu = \begin{cases} |\sigma(x)| & \text{if } \nu \text{ is real, corresponding to } \sigma : k \to \mathbb{R} \\ |\sigma(x)|^2 & \text{if } \nu \text{ is complex, corresponding to } \sigma, \overline{\sigma} : k \to \mathbb{C} \\ (N_k/Q)^{-\text{ord}_P} & \text{if } \nu \text{ is non-Arch., corresponding to } \mathcal{P} \subset \mathbb{R}_k \end{cases}
\]

for \( x \neq 0 \in k \). We also define \( \|0\|_\nu = 0 \). As we noted, these are not necessarily genuine absolute values. However, instead of having the triangle inequality, we have a value such that if \( a_1, \ldots, a_n \in k \), then

\[
\left\| \sum_{i=1}^{n} a_i \right\|_\nu \leq n^{N_\nu} \max_{1 \leq i \leq n} \|a_i\|_\nu,
\]

where

\[
N_\nu = \begin{cases} 1 & \text{if } \nu \text{ is real} \\ 2 & \text{if } \nu \text{ is complex} \\ 0 & \text{if } \nu \text{ is non-Archimedean}.\end{cases}
\]

If \( L \) is a finite extension of \( k \), \( \nu \in M_k \), and \( x \in k \), then

\[
\prod_{w \in M_L, w|\nu} \| x \|_w = \| x \|^{[L:k]}_\nu.
\]

Artin-Whaples extended the product formula on \( \mathbb{Q} \) to the number fields.

**Theorem 4 (Product Formula)** Let \( k \) be a number field. Let \( M_k \) be the canonical set of non-equivalent places on \( k \). Then, for every \( x \in k \) with \( x \neq 0 \),

\[
\prod_{\nu \in M_k} \| x \|_\nu = 1.
\]
0.2 Roth’s Theorem

Roth’s theorem was extended by Mahler to number field $k$ as follows:

**Theorem 5 (Roth)** Given $\epsilon > 0$, a finite set of places $S$ of $k$ containing $M_k^\infty$, and $\alpha_v \in \overline{Q}$ for each $v \in S$. Then for all, except for finitely many, $x \in k$,

$$\frac{1}{[k : Q]} \sum_{v \in S} -\log \min(\|x - \alpha_v\|_v, 1) \leq (2 + \epsilon)h(x),$$  \hspace{1cm} (9)

where $h(x)$ is the absolute logarithmic height defined by

$$h(x) = \frac{1}{[k : Q]} \sum_{v \in M_k} \log^+ \|x\|_v.$$  \hspace{1cm} (10)

Fix a finite set $S$ containing $M_k^\infty$, we define, for $a, x \in k$,

$$m(x, a) = \frac{1}{[k : Q]} \sum_{v \in S} \log^+ \frac{1}{\|x - a\|_v},$$  \hspace{1cm} (11)

$$N(x, a) = \frac{1}{[k : Q]} \sum_{v \in S} \log^+ \frac{1}{\|x - a\|_v}.$$  \hspace{1cm} (12)

Then the product formula (Theorem 4) reads

**Theorem 6** For all $x \in k^*$, $a \in k$

$$m(x, a) + N(x, a) = h(x) + O(1).$$

Theorem B1.2.5 can be restated as

**Theorem 7 (Roth)** Given $\epsilon > 0$, a finite set $S \subset M_k$ containing $M_k^\infty$, and distinct points $a_1, \ldots, a_q \in k$. Then the inequality

$$\sum_{j=1}^{q} m(x, a_j) \leq (2 + \epsilon)h(x)$$

holds for all, except for finitely many, $x \in k$.

Lang made the following conjecture with a more precise error term.
Conjecture 8 (Lang) Given $\epsilon > 0$, a finite set $S \subset M_k$ containing $M_k^\infty$, and distinct points $a_1, \ldots, a_q \in k$, the inequality

$$(q - 2)h(x) \leq \sum_{j=1}^{q} N^{(1)}(x, a_j) + (1 + \epsilon) \log h(x)$$

holds for all, except for finitely many, $x \in k$.

Roth’s theorem implies the following analogy of Picard’s Theorem.

Theorem 9 Let $k$ be a number field, and let $a_1, \ldots, a_q$ be distinct numbers in $k \cup \{\infty\}$. If $q \geq 3$, then there are only finitely many elements $x \in k$ such that $1/(x - a_j)$ (or $x$ itself if $a_j = \infty$) is an algebraic integer for all $1 \leq j \leq q$.

To further explore the analogy, we introduce more notation. Recall the Nevanlinna counting function for a meromorphic function $f$ is defined by

$$N_f(r, a) = \sum_{z \in D(r), z \neq 0} \text{ord}_z^+(f - a) \log \frac{r}{|z|} + \text{ord}_z^-(f - a) \log r.$$  

On the other hand, take $S = M_k^\infty$, then the number theoretic counting function $N(x, a)$ defined by (12) can be rewritten as

$$N(x, a) = \frac{1}{[k : Q]} \sum_{v \in M_k^\infty} \log^+ \frac{1}{\|x - a\|_v}$$

$$= \frac{1}{[k : Q]} \sum_{P \in \mathcal{R}_k} \text{ord}_P^+(x - a) \log(N_{k/Q} P) \quad (13)$$

where $\text{ord}_P^+ x = \max\{0, \text{ord}_P x\}$. So $N_f(r, a)$ and $N(x, a)$ can be compared by replacing $\log(r/|z|)$ in the definition of $N_f(r, a)$ with $\log(N_{k/Q} P)$ in the definition of $N(x, a)$. From this point of view, Paul Vojta has compiled a dictionary to translate the terms in Nevanlinna theory to the terms in Diophantine approximation. It is provided on p. 32.
**Nevanlinna Theory**
non-constant meromorphic function \( f \)
A radius \( r \)
A finite measure set \( E \) of radii
An angle \( \theta \)
\( |f(re^{i\theta})| \)
\( (\text{ord}_x f) \log \frac{r}{|x|} \)
Proximity function
\( m_f(r, a) = \int_0^{2\pi} \log^+ \left| \frac{1}{f(re^{i\theta}) - a} \right| \frac{d\theta}{2\pi} \)
Counting function:
\( N_f(r, a) = \text{ord}_0^+ (f - a) \log r + \sum_{0 < |z| < r} \text{ord}_z^+ (f - a) \log \frac{r}{|z|} \)
Characteristic function
\( T_f(r) = \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} \)
\( + N_f(r, \infty) \)
Jensen’s formula:
\( \int_0^{2\pi} \log |f(re^{i\theta})| \frac{d\theta}{2\pi} = N_f(r, 0) - N_f(r, \infty) + O(1) \)
First Main Theorem:
\( m_f(r, a) + N_f(r, a) = T_f(r) + O(1) \)
Weaker Second Main Theorem:
\( (q - 2)T_f(r) - \sum_{j=1}^{q} N_f(r, a_j) \leq \epsilon T_f(r) \)
Second Main Theorem:
\( (q - 2)T_f(r) - \sum_{j=1}^{q} N_f^{(1)}(r, a_j) \leq (1 + \epsilon) \log T_f(r) \)

**Diophantine Approximation**
infinite \( \{x\} \) in a number field \( k \)
An element of \( k \)
A finite subset of \( \{x\} \)
An embedding \( \sigma : k \rightarrow \mathbb{C} \)
\( |x|_\sigma \)
\( (\text{ord}_P x) \log(N_{k/Q} P) \)
Proximity function
\( m(x, a) = \sum_{\sigma : k \rightarrow \mathbb{C}} \log^+ \|x\|_\sigma \)
Counting function:
\( N(x, a) = \frac{1}{[k:Q]} \sum_{P \subset \mathbb{R}_k} \text{ord}_P^+(x - a) \log(N_{k/Q} P) \)
Logarithmic height
\( h(x) = \frac{1}{[k:Q]} \sum_{\sigma : k \rightarrow \mathbb{C}} \log^+ \|x\|_\sigma \)
\( + N(x, \infty) \)
Atin-Whaples Product Formula:
\( \sum_{\sigma : k \rightarrow \mathbb{C}} \log \|x\|_\sigma = N(x, 0) - N(x, \infty) \)
Height Property:
\( m(x, a) + N(x, a) = h(x) + O(1) \)
Roth’s Theorem:
\( (q - 2)h(x) - \sum_{j=1}^{q} N(x, a_j) \leq \epsilon h(x) \)
Lang’s conjecture:
\( (q - 2)h(x) - \sum_{j=1}^{q} N^{(1)}(x, a_j) \leq (1 + \epsilon) \log h(x) \)

Note that, in above, we use the notation \( \leq \) to denote that the inequality holds for all \( r \) except a set \( E \subset (0, + \infty) \) with finite Lebesgue measure in Nevanlinna theory and the inequality holds for all, except for finitely many, \( x \in k \) in Diophantine approximation.