VECTOR BUNDLES

PIT-MANN WONG*

1. COMPLEX LINE BUNDLES

Let L be a smooth complex line bundle over a differentiable manifold M. Locally the bundle is trivial. Thus there exists an open cover of $\mathcal{U} = \{U\}$ so that $M|_U$ is trivial when restrict to $U \in \mathcal{U}$. this means that we have a bundle isomorphism:

$$\begin{array}{ccc} L|U & \xrightarrow{\phi_U} & U \times \mathbb{C} \\ \pi \downarrow & & \downarrow \mathrm{pr}_1 \\ U & = & U \end{array}$$

and on the overlap of two such trivializing open sets U and V the bundle isomorphism

$$\phi_U \circ \phi_V^{-1} : (U \cap V) \times \mathbb{C} \to (U \cap V) \times \mathbb{C}$$

may be expressed as $\phi_U \circ \phi_V^{-1}(z; v) = (z, g_{UV}(z)v)$ where g_{UV} is a smooth non-vanishing function on $U \cap V$. The functions $\{g_{UV}\}$ will be referred to as transition functions. These functions clearly satisfies the conditions:

(a) $g_{UU} = \mathrm{Id}$,

(b) $g_{UV} = 1/g_{UV}$,

(c) (multiplicative cocycle condition) $g_{UV}g_{VW}g_{WV} = 1$.

A cocylce defines an element in the group $H^1(M, \mathcal{C}^*)$ where $\mathcal{C}^*$ is the sheaf of germs of non-vanishing smooth functions. The trivial line bundle corresponds to a cocycle $\{g_{UV}$ satisfying $g_{UV} = g_U g_V^{-1}$ where $\{g_U\}$ is a set of non-vanishing smooth functions on $U \in \mathcal{U}$. Thus the isomorphism classes of complex bundles are identified with the multiplicative group $H^1(M, \mathcal{C}^*)$. The transition functions of the inverse L^{-1} of a line bundle L with transition functions $\{g_{UV}\}$ is given by $\{1/g_{UV}\}$. The transition functions of the tensor product of two line bundles L and $\tilde{L}$ is the product $\{g_{UV}\tilde{g}_{UV}\}$ of the transition functions of the two bundles.

The short exact sequence of sheaves

$$0 \to \mathbb{Z} \longrightarrow \mathcal{C} \stackrel{e^{2\pi\sqrt{-1}}}{\longrightarrow} \mathcal{C}^* \to 0$$

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induces a long exact sequence of cohomology groups:

$$\to H^1(M,\mathcal{C}) \to H^1(M,\mathcal{C}^*) \xrightarrow{[c_1]} H^2(M,\mathbb{Z}) \to H^2(M,\mathcal{C}) \to$$

where the connecting map $[c_1]$ is the homomorphism sending a line bundle to its Chern class. The fact that $\mathcal{C}$ is a fine sheaf implies that $H^i(M,\mathcal{C}) = 0$ for all $i \geq 1$. This can be verified directly as follows. Let ρ_k be a partition of unity subordinate to an open cover U_k of M. An element of $H^1(M,\mathcal{C})$ is represented by an additive ($\mathcal{C}$ is a sheaf of additive groups) cocycle $\{f_{ij}\}$ (this means that $f_{ij} + f_{jk} + f_{ki} = 0$). The smooth function on U_i defined

$$f_i = \sum_k \rho_k f_{ik}$$

satisfies the condition that $f_j - f_i = f_{ij}$. In other words, $\{f_{ij}\}$ is a coboundary. The proof of the higher cohomology groups is similar.

Thus we have

Proposition 1.1. *The map $[c_1] : H^1(M,\mathcal{C}^*) \to H^2(M,\mathbb{Z})$ is an isomorphism.*

The map $[c_1]$ can be described explicitly as follows. Let $\{g_{ij} \in H^0(U_i \cap U_j, \mathcal{C}^*)\}$ be a cocyle. Then there exists $f_ij \in H^0(U_i \cap U_j, \mathcal{C})$ such that $\exp 2\pi\sqrt{-1} f_{ij} = g_{ij}$. The cocycle condition on g_{ij} implies that $a_{ijk} = f_{jk} - f_{ik} + f_{ij}$ is an integer. Moreover, $a_{jk\ell} - a_{ik\ell} + a_{ij\ell} - a_{ijk} = 0$ hence a_{ijk} is a cocycle and defines a class in $H^2(M,\mathbb{Z})$ and we define

$$[c_1(\{g_{ij}\})] = [a_{ijk}].$$

Conversely, starting from a cocycle a_{ijk}, define

$$f_{ij} = \sum_k \rho_k a_{ijk}, \quad g_{ij} = \exp 2\pi\sqrt{-1} f_{ij}.$$

Then g_{ij} satisfies the cocycle condition. The map,

$$[a_{ijk}] \mapsto [g_{ij}]$$

just defined is easily seen to be the inverse of the map $[c_1]$.

We may inject $H^2(M,\mathbb{Z})$ in $H^2(M,\mathbb{R})$ and identify the later with the deRham cohomology group $H^2_{\text{deRham}}(M) = \{\text{closed } 2-\text{forms}\}/\{\text{exact } 2-\text{forms}\}$. Thus we may express $[c_1(\{g_{ij}\})] = [a_{ijk}]$ as a 2-form by spelling out the deRham isomorphism. First we define 1-forms

$$\omega_i = \sum_k \rho_k df_{ik}.$$

It is easily checked that

$$\omega_i - \omega_j = \frac{dg_{ij}}{g_{ij}}.$$

A collection of 1-forms satisfying this condition is known as connection forms for the bundle g_{ij}. To get 2-forms we take the exterior derivative:

$$\Omega_i = d\omega_i$$

and we check that

$$\Omega_i = g_{ij}\Omega_j g_{ij}^{-1}.$$

This is the curvature form of the bundle g_{ij}.

Consider now the case of a holomorphic bundle L. In this case the transition function g_{ij} is a non-vanishing holomorphic function on U_{ij} and so defines an element of $H^1(M, \mathcal{O}^*)$. We still have a short exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

and the associated long exact sequence

$$\to H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}^*) \xrightarrow{[c_1]} H^2(M, \mathbb{Z}) \to H^2(M, \mathcal{O}) \to$$

The main difference is that we do not have vanishing of $H^1(M, \mathcal{O})$ and $H^2(M, \mathcal{O})$ in general. We can express the Chern class via a Hermitian metric along the fibers. Local trivialization allows us to express a holomorphic section σ of L via a collection of local holomorphic functions s_i on U_i satisfying the condition

$$s_i = g_{ij} s_j.$$

Let $|\ |$ be the usual absolute value on $\mathbb{C}$ then a Hermitian fiber metric h on L is identified over U_i with $h_i |\ |^2$ where h_i is a positive function on U_i. Thus, over $U_i \cap U_j$ we have

$$h_i |s_i|^2 = h_j |s_j|^2 = h_j |g_{ij}|^{-2} |s_j|^2.$$

Thus the metric $h4$ is identified with the collection of positive functions h_i satisfying the transition condition $h_j = |g_{ij}|^2 h_i$. The 1-forms $\omega_i = \partial \log h_i$ satisfies the condition

$$\omega_i - \omega_j = \partial \log \frac{h_i}{h_j} = \partial \log |g_{ij}|^2 = \frac{\partial g_{ij}}{g_{ij}}.$$

These 1-forms is the hermitian connection form of the metric h. Usually this is written in shorthand as $\omega = \partial \log h$. The Hermitian curvature form is $\Omega = \bar{\partial}\omega$. Locally this is represented by a collection of 2-forms $\Omega_i = \bar{\partial}\partial \log h_i$. the first Chern form $c_1(L, h)$ is by definition

$$\frac{\sqrt{-1}}{2\pi} \bar{\partial}\partial \log h_i = -dd^c \log h_i$$

where

$$d^c = \frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial).$$

Example 1.1. Let (L, h) be a Hermitian holomorphic line bundle over a compact (without boundary) Riemann surface M and assume that there exists a non-trivial holomorphic section σ of L. The zero set $[s = 0]$ consists of a finite number of points $a_1, ..., a_N$ with vanishing order $\mu_1, ..., \mu_N$ respectively. Choose local coordinate discs Δ_i centered at each a_i. At a point $a \neq a_i, i = 1, ..., N$ there exists local coordinate neighborhood U of a and we may write $||\sigma||^2 = h_U|s_U|^2$ with s_U non-vanishing and holomorphic on U. Thus $\log |s_U|^2$ is harmonic, so that $dd^c \log ||\sigma||_h^2 = dd^c \log h_U + dd^c \log |s_U|^2 = -c_1(L, h)|_U$. From this we deduce that

$$\int_M dd^c[\log ||\sigma||_h^2] = \int_{M\setminus\cup_i\Delta_i} dd^c \log ||s||_h^2 + \sum_i \Big(\int_{\Delta_i} dd^c \log h_i + \int_{\Delta_i} dd^c \log[|s_i|^2] \Big)$$
$$= -\int_M c_1(L, h) + \sum_i \int_{\Delta_i} dd^c[\log |s_i|^2].$$

The notation $dd^c[\ ]$ indicates that differentiation is taken in the sense of distribution. On Δ_i we may write $s_i = z^{\mu_i}\phi_i$ where ϕ_i is holomorphic and non-vanishing. Thus $dd^c[\log |s_i|^2] = \mu_i dd^c \log[|z - a_i|^2]$ and $dd^c[\log |z - a_i|^2]$ is the delta function at a_i. This implies that

$$\int_M dd^c[\log ||\sigma||_h^2] = -\int_M c_1(L, h) + \#\text{ of zeros, counting multiplicities, of } \sigma.$$

On the other hand, since M is compact without boundary, Stokes Theorem which applies to currents implies that

$$\int_M dd^c[\log ||\sigma||_h^2] = 0.$$

We arrive at the identity

$$\int_M c_1(L, h) = \#\text{ of zeros, counting multiplicities, of } \sigma.$$

An analogous argument applied to a meromorphic section yields

Theorem. *Let σ be a meromorphic section of a holomorphic line bundle over a compact, without boundary, Riemann surface M. Then*

$$\int_M c_1(L, h) = \#([\sigma = 0]) - \#([\sigma = \infty])$$

where $\#([\sigma = 0])$ is the number of zeros, counting multiplicities, of σ and $\#([\sigma = \infty])$ is the number of poles, counting multiplicities, of σ.

Remark. (a) Observe that the non-negativity (positivity) of Chern number is a necessary condition for the existence of global holomorphic sections (with zeros).

(b) Taking L to be the trivial line bundle, we get

Corollary. *Let f be a meromorphic function on a compact, without boundary, Riemann surface M. Then the number of zeros, counting multiplicities, of f and the number of poles, counting multiplicities, of f are equal.*

(c) In the literature, the preceding theorem is often written simply as an identity of current: $c_1(L) = \operatorname{div} \sigma$.

Theorem 1.1 (Kodaira)**.** *Let L be an ample line bundle over a projective variety M. Then for any coherent sheaf $\mathcal{S}$ over M there exists a positive integer m_0 such that $H^q(M, \mathcal{S} \otimes L^m) = 0$ for all $q \geq 1$ and $m \geq m_0$.*

In particular, for any ample line bundle, $H^q(M, \mathcal{S} \otimes L^m) = 0$ for all $q \geq 1$ and for all m sufficiently large. This can be made precise (so long as the underlying field is algebraically closed and of characteristic zero):

Theorem 1.2 (Kodaira)**.** *Let L be an ample line bundle over a projective variety M. Then $H^q(M, K_M \otimes L) = 0$ for all $q \geq 1$, where $K_M = \det T^*M$ is the canonical bundle. Thus, if $K_M^{-1} \otimes L^m$ is ample then $H^q(M, L^m) = 0$ for all $q \geq 1$.*

Theorem 1.3 (Serre Duality)**.** *Let M be a compact complex manifold and V a holomorphic vector bundle over M. Then*

$$\dim H^q(M, V) = \dim H^{n-q}(M, V^* \otimes K_M), \quad n = \dim_{\mathbb{C}} M$$

where V^* is the dual of V and K_M is the canonical bundle of M

Theorem 1.4 (Akizuki-Nakano)**.** *Let L be an ample line bundle over a projective variety M. Then $H^q(M, \Omega_M^p \otimes L) = 0$ for all $p + q > \dim_{\mathbb{C}} M$, where where $\Omega_{\mathbb{P}^n}^p$ is the sheaf of germs of holomorphic p-forms.*

Remark 1.1. *By Serre's Duality Theorem we get, for an ample line bundle L,*
- *$H^q(M, L^{-1}) = 0$ for all $q \leq \dim M - 1$ and*
- *$H^q(M, \Omega_M^p \otimes L^{-1}) = 0$ for all $p + q < \dim_{\mathbb{C}} M$.*

Example 1.2. The following formulas (Bott) are well-known:

$$h^q(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^p(k)) = \begin{cases} 1, & k = 0, p = q \\ C_k^{k+n-p} C_p^{k-1}, & q = 0 \\ C_{-k}^{-k+p} C_{n-p}^{-k-1}, & q = n \\ 0, & \text{otherwise} \end{cases}$$

where $\Omega_{\mathbb{P}^n}^p$ is the sheaf of germs of holomorphic p-forms and $\Omega_{\mathbb{P}^n}^p(k) = \Omega_{\mathbb{P}^n}^p \otimes \mathcal{O}_{\mathbb{P}^n}(k)$, $\mathcal{O}_{\mathbb{P}^n}(1)$ (resp. $\mathcal{O}_{\mathbb{P}^n}(-1)$) is the hyperplane line bundle (resp., the tautological line bundle) and $\mathcal{O}_{\mathbb{P}^n}(k)$ is k-fold tensor product of the hyperplane line bundle or the tautological line bundle. In particular, we have

$$h^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = \begin{cases} C_k^{k+n}, & q = 0 \\ C_n^{-k-1}, & q = n \\ 0, & \text{otherwise.} \end{cases}$$

Take $k = 0$, we have:

$$h^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 0 \quad q \geq 1.$$

This implies that $H^1(\mathbb{P}^n, \mathcal{O}^*) \cong H^2(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}$. Observe also that for the range $1 \leq q \leq n-1$

$$h^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(k)) = \begin{cases} 0, & \text{if } p \neq q, \\ 1, & \text{if } p = q \end{cases}$$

for all k.

Let (X, ω) be a compact symplectic manifold and assume that

$$[\omega] \in H^2(M, \mathbb{Z}).$$

There exists a Hermitian line bundle (L, h) over M such that the curvature $\Theta = \Theta^h$ of the Hermitian connection $\nabla = \nabla^h$ satisfies

$$\omega = c_1(L, h) = \frac{\sqrt{-1}}{2\pi}\Theta^h.$$

Let J be an almost complex structure on M such that

$$\omega(X, JX) > 0 \;\text{ for }\; X \neq 0 \;\text{ and }\; \omega(JX, JY) = \omega(X, Y).$$

The manifold is equipped with the Riemannian metric $g(X, Y) = \omega(X, JY)$. The J-holomorphic (resp. J-anti-holomorphic) tangent bundle is denoted by $T^{1,0}M$ (resp. $T^{0,1}M$). The almost complex structure J on TM is identified with $\sqrt{-1}\,\mathrm{Id}$ (resp. $-\sqrt{-1}\,\mathrm{Id}$) on $T^{1,0}M$ (resp. $T^{0,1}M$).

Definition 1.1. *Let (M, ω, J, g, L, h) be as above. (a) A sequence of sections, σ_k, of L^k is said to be asymptotically J-holomorphic if the following conditions and satisfied*

$$\begin{cases} |\sigma_k| & = O(1), \\ |\bar{\partial}_J \sigma_k| & = O(1), \\ |\nabla \sigma_k| & = O(k^{1/2}), \\ |\nabla \bar{\partial}_J \sigma_k| & = O(k^{1/2}), \\ |\nabla\nabla \sigma_k| & = O(k) \end{cases}$$

where the norms are induced by h and g.

(b) A sequence of sections, σ_k, of L^k is said to be transversal to the zero section, or simply transversal, if there exists $\epsilon > 0$ such that $|\nabla \sigma_k| > \epsilon k^{1/2}$ on the set $\{x \in M \mid |\sigma_k(x)| < \epsilon\}$.

Theorem 1.5 (Donaldson)**.** *Let (M, ω, J, g, L, h) be as above. Then there exists sections of L^k which are asymptotically J-holomorphic and transversal.*

Corollary 1.1 (Donaldson)**.** *The zero sets of asymptotically J-holomorphic transversal sections of L^k are embedded symplectic submanifolds.*

Let (E, h_E) be any Hermitian vector bundle over M. The concept of asymptotically holomorphic sections of $E \otimes L^k$ is analogously defined as above:

Definition 1.2. *Let $(M, \omega, J, g, L, h, E, h_E)$ be as above. A sequence of sections, σ_k, of $E \otimes L^k$ is said to be asymptotically J-holomorphic if the following conditions and satisfied*

$$\begin{cases} |\sigma_k| & = O(1), \\ |\bar{\partial}_J \sigma_k| & = O(1), \\ |\nabla \sigma_k| & = O(k^{1/2}), \\ |\nabla \bar{\partial}_J \sigma_k| & = O(k^{1/2}), \\ |\nabla\nabla \sigma_k| & = O(k) \end{cases}$$

where the norms are induced by h, h_E and g.

The concept of transversality is slightly more complicated

Definition 1.3. *Let $(M, \omega, J, g, L, h, E, h_E)$ be as above. A sequence of sections, σ_k, of $E \otimes L^k$ is said to be transversal to the zero section (or simply, transversal) if there exists $\epsilon > 0$ such that the map*

$$\nabla \sigma_k : T_x M \to E \otimes E \otimes L_x^k$$

is surjective and admits a right inverse with norm smaller than $\epsilon^{-1} k^{-1/2}$ on the set $\{x \in M \mid |\sigma_k(x)| < \epsilon\}$.

Theorem 1.6 (Auroux)**.** *Let $(M, \omega, J, g, L, h, E, h_E)$ be as above. Then there exists sections of $E \otimes L^k$ which are asymptotically J-holomorphic and transversal.*

Corollary 1.2 (Auroux)**.** *The zero sets of asymptotically J-holomorphic transversal sections of $E \otimes L^k$ are embedded symplectic submanifolds.*

Denote by $H^0(M, L^k)$ the space of asymptotical J-holomorphic transversal sections. Let $\phi_0, \phi_1, ..., \phi_{N_k}$ be a basis of $H^0(M, L^k)$ and define the map

$$\Phi_k : M \to \mathbb{P}^{N_k}, \quad \Phi_k = [\phi_0, \phi_1, ..., \phi_{N_k}].$$

The following analogue of Kodaira's Embedding Theorem:

Theorem 1.7 (Shiffman-Zelditch). *Let (M,ω,J,g,L,h) be as above. Then, for k sufficiently large, the map $\Phi_k : M \to \mathbb{P}^{N_k}$, defined by the space of asymptotical J-holomorphic transversal sections, is an embedding. In fact Φ_k is an almost isometry in the sense that*

$$||\frac{1}{N_k}\Phi^*\omega_{\mathrm{FS},k} - \omega||_{\mathcal{C}^k} = O(\frac{1}{N_k})$$

where $\omega_{\mathrm{FS},k}$ is the Fubini-Study metric on $\mathbb{P}^{N_k}$.

2. Curvature of vector bundles

Let (E,h) be a holomorphic Hermitian vector bundle of rank (over $\mathbb{C}$) r over a complex manifold M of complex dimension n. A *connection* is a $\mathbb{C}$-linear map

$$\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E) \tag{2.1}$$

where $\mathcal{A}^p(E)$ is the space of (local) smooth p-forms with values in E satisfying the conditions:

(*i*) (Leibnitz' rule) $\nabla(fs) = df \otimes s + f\nabla s$ *for all local smooth function f and local smooth section of $\mathcal{A}^0(E)$.*

A connection is said to be *hermitian* if the next two conditions are satisfied:

(*ii*) (complex structure compatibility) *let $\mathcal{A}^1(E) = \mathcal{A}^{(1,0)}(E) \oplus \mathcal{A}^{(0,1)}(E)$ be the decomposition of E-valued 1-forms into type $(1,0)$ and $(0,1)$ and let $\nabla = \nabla' + \nabla''$ where $\nabla' = \pi' \circ \nabla$ and $\nabla'' = \pi'' \circ \nabla$, $\pi' : \mathcal{A}^1(E) \to \mathcal{A}^{(1,0)}(E)$ and $\pi'' : \mathcal{A}^1(E) \to \mathcal{A}^{(0,1)}(E)$ being the natural projections, then $\nabla'' = \overline{\partial}$;*

(*iii*) (metric compatibility) $d\langle\sigma,\tau\rangle_h = \langle\nabla\sigma,\tau\rangle_h + \langle\sigma,\nabla\tau\rangle_h$.

If $e = \{e_1,...,e_r\}$ is a local frame of E with dual frame $e^* = \{e_1^*,...,e_r^*\}$. By definition,

$$\nabla e_\alpha = \sum_\beta \theta_\alpha^\beta \otimes e_\beta \tag{2.2}$$

where $\theta_e = (\theta_\alpha^\beta)_{1\le\alpha,\beta\le r}$ are 1-forms, hneceforth referred to as the matrix of connection 1-forms relative to the frame e. A section of E is of the form $s = \sum_{\alpha=1}^r s^\alpha \otimes e_\alpha$ and Leibnitz' rule implies that

$$\nabla s = \sum_{\alpha=1}^r ds^\alpha \otimes e_\alpha + \sum_{1\le\alpha,\beta\le r} s^\beta\theta_\beta^\alpha \otimes e_\alpha.$$

Given a vector field v on M define ∇_v by

$$\nabla_v s = \sum_{\alpha=1}^r ds^\alpha(v)\, e_\alpha + \sum_{1\le\alpha,\beta\le r} \theta_\beta^\alpha(v)\, s^\beta e_\alpha \tag{2.3}$$

then $\nabla_v : \mathcal{A}^0(E) \to \mathcal{A}^0(E)$ is linear over $\mathbf{C}$. On the other hand, if a, b are *functions* and v, w are vector fields then

$$\nabla_{av+bw}s = a\nabla_v s + b\nabla_w s.$$

Suppose that $e^{'} = \{e_1^{'}, ..., e_r^{'}\}$ is another local frame of E then there is a non-singular transformation $g = (g_i^j)_{1\leq i,j\leq r}$ of (local) smooth functions such that $e' = ge$ more precisely,

$$e_i' = \sum_{j=1}^{r} g_i^j e_j \tag{2.4}$$

(or simply $e^{'} = ge$) and the corresponding connection forms are related by the formula:

$$\theta_{e'} = g\theta_e g^{-1} + (dg)g^{-1}. \tag{2.5}$$

Conversely given an open cover $\{U\}$ of M with local frames $\{e_U\}$ of E and suppose that there exists matrices of 1-forms, $\{\theta_U\}$, over $\{U\}$ satisfying the transformation law (2.5) on the overlaps. Let ∇_U be defined as in (2.2) then $\nabla_U = \nabla_{U'}$ on $U \cap U'$ hence $\nabla = \nabla_U$ on U is a globally well-defined connection.

The components of the metric h relative to a local frame are denoted by

$$h_{i\overline{j}} = < e_i, e_j >_h, \;\; 1 \leq i, j \leq r.$$

The transformation law for the metric is easily seen to be:

$$h_{i\overline{j}} = g_i^k h_{k\overline{l}}^{'} g_{\overline{j}}^{\overline{l}} \tag{2.6}$$

where $g = (g_i^k)$ is the transition matrix of the frames e' and e (see (2.4)) and we write $g_{\overline{j}}^{\overline{l}}$ for $\overline{g_k^l}$. The preceding transformation law will be simply expressed as $h = gh^{'}\overline{g}^t$, the superscript t indicates that we are taking the transpose of the matrix.

Proposition 2.1. *Given a Hermitian holomorphic vector bundle* (E, h) *a hermitian connection exists and is unique. In fact the connection matrix is given by*

$$\theta = (\partial h)h^{-1},$$

more precisely, in terms of a local holomorphic frame $e_1, ..., e_r$,

$$\theta_\alpha^\gamma = \sum_{\beta=1}^{r} (\partial h_{\alpha\overline{\beta}})h^{\overline{\beta}\gamma}$$

where $h_{\alpha\overline{\beta}} = < e_\alpha, e_\beta >_h$ *and* $(h^{\overline{\beta}\gamma})_{1\leq\gamma,\beta\leq r}$ *is the inverse of the Hermitian matrix* $(h_{\alpha\overline{\beta}})_{1\leq\alpha,\beta\leq r}$.

Proof. For the proof of the Proposition we choose an holomorphic frame e then since a hermitian connection ∇ is compatible with the complex structure of M (i. e., $\nabla^{''} = \overline{\partial}$) we have

$$0 = \overline{\partial} e_\alpha = \nabla^{''} e_\alpha = (\theta_\alpha^\beta)^{0,1} \otimes e_\beta$$

for all α from which we infer easily that, $(\theta_\alpha^\beta)^{0,1}$, the $(0,1)$-component of θ_α^β vanishes for all α and β, i.e., θ is of type (1, 0). From the metric compatibility:

$$dh_{\alpha\overline{\beta}} = \theta_\alpha^\gamma h_{\gamma\overline{\beta}} + \theta_{\overline{\beta}}^{\overline{\lambda}} h_{\alpha\overline{\lambda}}.$$

We conclude, by comparing types, that

$$\partial h_{\alpha\overline{\beta}} = \theta_\alpha^\gamma h_{\gamma\overline{\beta}},\ \overline{\partial} h_{\alpha\overline{\beta}} = \theta_{\overline{\beta}}^{\overline{\lambda}} h_{\alpha\overline{\lambda}}.$$

In fact these two conditions are equivalent (via conjugation) because the metric is hermitian (i.e., $\overline{h_{\alpha\overline{\lambda}}} = h_{\lambda\overline{\alpha}}$). This condition is abbreviated simply by $\theta = (\partial h)h^{-1}$. We must check that θ is a well-defined connection, i.e., (2.5) is satisfied. This follows easily from the transformation law (2.6): $h = gh^{'}\overline{g}^t$ for the metric (where $e^{'}$ is another holomorphic local frame so $e^{'} = ge$ with g holomorphic):

$$(\partial h)h^{-1} = \{(dg)h^{'}\overline{g}^t + g\partial h^{'}\overline{g}^t\}\{(\overline{g}^t)^{-1}(h^{'})^{-1}g^{-1}\} = (dg)g^{-1} + g(\partial h^{'})(h^{'})^{-1}g^{-1}$$

as claimed. □

Proposition 2.2. *Let* (E, h) *be a hermitian holomorphic vector bundle with Hermitian connection* ∇ *then* (i) *with respect to a local holomorphic frame the curvature forms* $\theta = (\theta_\alpha^\beta)$ *are of type* $(1,0)$; (ii) *with respect to a local unitary frame the connection matrix is skew-hermitian, i.e.,*

$$\theta_\alpha^\beta + \overline{\theta_\beta^\alpha} = 0.$$

Proof. Assertion (i) follows immediately from condition (ii) (compatibility with the complex structure on M) of a hermitian connection. Condition (iii) (compatibility with the metric) on the other hand, implies that,

$$0 = d\langle e_\alpha, e_\beta\rangle_h = \langle \theta_\alpha^\gamma e_\gamma, e_\beta\rangle_h + \langle e_\alpha, \theta_\beta^\mu e_\mu\rangle_h = \theta_\alpha^\beta + \overline{\theta_\beta^\alpha}$$

as claimed. □

Given a connection $\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ define a differential operator $D : \mathcal{A}^1(E) \to \mathcal{A}^2(E)$ as follows. A section ϕ of $\mathcal{A}^1(E)$ is locally of the form $\phi^\alpha \otimes e_\alpha$ where ϕ^α is a local 1-form then

$$D\phi = d\phi^\alpha \otimes e_\alpha - \phi^\alpha \wedge \nabla e_\alpha = d\phi^\alpha \otimes e_\alpha - \phi^\alpha \wedge \theta_\alpha^\beta \otimes e_\beta. \tag{2.7}$$

Just as the case of ∇, D is also decomposed into operators $D^{'}$ and $D^{''}$, specifically

$$D^{'}\phi = \partial\phi^\alpha - \phi^\alpha \wedge \nabla^{'} e_\alpha,\ D^{''}\phi = \overline{\partial}\phi^\alpha - \phi^\alpha \wedge \nabla^{''} e_\alpha.$$

It is clear that if ∇ is hermitian then $D^{''} = \overline{\partial}$. The curvature operator is, by definition,

$$D\nabla = D \circ \nabla : \mathcal{A}^0(E) \to \mathcal{A}^2(E). \tag{2.8}$$

If $\nabla$is the hermitian connection of (E, h) is hermitian the curvature is referred to as *the hermitian curvature of* h.

Proposition 2.3. *The curvature $D\nabla$ is a tensor, i.e., it is linear over the sheaf of smooth functions (not merely over* $\mathbf{C}$*), i. e.,*

$$D\nabla(fs) = fD\nabla s$$

for all local smooth functions f.

Proof. Let θ be the connection matrix with respect to a local frame then

$$D\nabla e_\alpha = D(\theta_\alpha^\beta \otimes e_\beta) = d\theta_\alpha^\beta \otimes e_\beta - \theta_\alpha^\beta \wedge \theta_\beta^\gamma \otimes e_\gamma$$

which is abbreviated as

$$\Theta = D\nabla = d\theta - \theta \wedge \theta. \tag{2.9}$$

We have $\nabla(fs) = (df) \otimes s + f\nabla s$ hence

$$D\nabla(fs) = -df \wedge \nabla s + df \wedge \nabla s + fD\nabla s = fD\nabla s.$$

□

The preceding Proposition implies that $D\nabla \in H^0(M, \wedge^2 T^*_{\mathbf{C}} M \otimes \mathrm{Hom}(E, E))$. The transformation law for the curvature forms relative to different local frames is much simpler than that for the connection forms, indeed, the property $D\nabla(fs) = fD\nabla s$ for any *smooth function* f implies that:

$$\Theta_{e'} = g\Theta_e g^{-1} \tag{2.10}$$

where g is the transformation between the local frames e' and e, i.e., $e = ge'$.

Proposition 2.4. *Let (E, h) be a Hermitian holomorphic vector bundle and let ∇ be the unique hermitian connection with curvature $D\nabla$. Then relative to any local frame the curvature matrix is skew-hermitian, i.e.,*

$$\Theta_\beta^\alpha + \overline{\Theta_\alpha^\beta}$$

and of type $(1, 1)$.

Proof. By Proposition 2.2 relative to a local *unitary* frame the connection matrix θ_β^α is skew hermitian hence

$$\overline{\Theta_\beta^\alpha} = \overline{d\theta_\beta^\alpha - \theta_\beta^\gamma \wedge \theta_\gamma^\alpha} = -d\theta_\alpha^\beta + \theta_\gamma^\beta \wedge \theta_\alpha^\gamma = -\Theta_\alpha^\beta.$$

This shows that the curvature matrix is also skew-hermitian. Note that $\nabla = \nabla' + \overline{\partial}$ hence $D\nabla = D'\nabla' + D'\overline{\partial} + \overline{\partial}\nabla'$ as $\overline{\partial}^2 = 0$. This implies that that the curvature has no $(0,2)$-component. The skew-hermitian property then implies that the $(2,0)$-component is zero as well. This shows that relative to a unitary frame the curvature forms are of type (1, 1). The transformation law (2.6) implies that the same is true for any local frame. □

The skew-hermitian property implies that $\sqrt{-1}\Omega_\alpha^\alpha = \overline{\sqrt{-1}\Omega_\alpha^\alpha}$, i.e.,

$$\sqrt{-1}\Omega_\alpha^\alpha \text{ is a real } (1,1)\text{-form for all } \alpha = 1, ..., r. \tag{2.11}$$

The property that the curvature is of type (1, 1) implies that

$$D\nabla = D^{'}\overline{\partial} + \overline{\partial}\nabla^{'} \text{ and } \Theta = \overline{\partial}\theta. \tag{2.12}$$

Proposition 2.5. *Let (E, h) be a hermitian holomorphic vector bundle over a complex manifold M. Then at any point $x_0 \in M$ there exists a local holomorphic frame $e_1, ..., e_r$ over an open neighborhood of x_0 which is unitary and parallel at x, i.e., $h_{i\bar{j}}(x_0) =< e_i, e_j >_h (x_0) = \delta_i^j$ and $dh_{i\bar{j}}(x_0) = 0$ for all $1 \leq i, j \leq r$. In particular, all connection forms relative to this local frame vanishes at x_0 and the curvature at x_0 is given by $(\overline{\partial}\partial h)(x_0)$.*

Proof. Since h is hermitian it is clear (by diagonalization and re-scaling) that there is a local holomorphic frame $e = (e_1, ..., e_r)$ over an open neighborhood U of x_0 such that $h_{i\bar{j}}(x_0) = \delta_i^j$ at the point x_0. Choose $(U, z = (z^1, ..., z^n)), n = \dim M$ to be a local coordinate neighborhood so that x_0 is the origin. Let $H(z) = I_r + A(z)$ where I_r is the $r \times r$ identity matrix and

$$A(z) = (\sum_{k=1}^{n} A_{ik}^j z^k)_{1 \leq i,j \leq r}, \;\; A_{ik}^j = -\frac{\partial h_{i\bar{j}}}{\partial z^k}(0).$$

Define a new frame $\tilde{e} = eH$. Denote by h_e the matrix $(< e_i, e_j >_h)_{1 \leq i,j \leq r}$ and $h_{\tilde{e}}$ the matrix $(< \tilde{e}_i, \tilde{e}_j >_h)_{1 \leq i,j \leq r}$ then

$$h_{\tilde{e}} = \overline{H}^t h_e H = (I_r + \overline{A})^t h_e (I_r + A).$$

Since $h_e = I_r$ and $A = 0_r$ (the $r \times r$ zero matrix) at 0 (i.e., the point x_0), we have:

$$h_{\tilde{e}} = I_r \text{ and } dh_{\tilde{e}} = d(\overline{A}^t) + dh_e + dA$$

at x_0. By construction $dA = -\partial h_e$ at x_0 and so, by the hermitian property of h, $d\overline{A}^t = -\overline{\partial} h_e$ at x_0. These imply that $dh_{\tilde{e}} = 0$ at x_0 and so $\tilde{e}$ is the required frame. The connection matrix $\theta_{\tilde{e}}$ relative to the frame $\tilde{e}$ is, by definition, $(\partial h_{\tilde{e}})h_{\tilde{e}}^{-1} = 0$ at x_0. This implies that, at the point x_0,

$$\Theta_{\tilde{e}} = d\theta_{\tilde{e}} - \theta_{\tilde{e}} \wedge \theta_{\tilde{e}} = d\theta_{\tilde{e}} = d((\partial h_{\tilde{e}})h_{\tilde{e}}^{-1}) = (\overline{\partial}\partial h_{\tilde{e}})h_{\tilde{e}}^{-1} - \partial h_{\tilde{e}} \wedge d\partial h_{\tilde{e}}^{-1}$$

hence $\Theta_{\tilde{e}} = \overline{\partial}\partial h_{\tilde{e}}$ as $h_{\tilde{e}}$ is the identity matrix and $\partial h_{h\tilde{e}} = 0$ at the point x_0. $\square$

A variation of the preceding Proposition is the following:

Proposition 2.6. *Let (E, h) be a hermitian holomorphic vector bundle. Then at any point $x \in M$ there exists a local unitary frame $e_1, ..., e_r$ over an open neighborhood of x which is parallel at x.*

The proof of the preceding Proposition is quite straightforward (via the Gram-Schmidt process) and is left to the reader.

Let $S \subset E$ be a holomorphic sub-bundle with quotient bundle $Q = E/S$. As a $\mathcal{C}^\infty$ bundle $Q \cong S^\perp$, the orthogonal (relative to a hermitian metric on E) complement of S. The bundles S and Q are naturally equipped with hermitian metrics $h|_S, h|_{S^\perp}$ induced by h respectively. Denote by ∇^E, ∇^S and ∇^Q the corresponding hermitian connections. The connection $\nabla_S : \mathcal{A}^0(S) \to \mathcal{A}^1(S)$ is not the restriction $\nabla^E|_{\mathcal{A}^0(S)} : \mathcal{A}^0(S) \to \mathcal{A}^1(E)$ but rather, the S-component of $\nabla^E|_{\mathcal{A}^0(S)}$, more precisely

$$\nabla^S = \pi_S \circ \nabla^E|_{\mathcal{A}^0(S)} \tag{2.13}$$

$\pi_S : \mathcal{A}^0(E) \to \mathcal{A}^0(S)$ being the natural projection induced by the isomorphism $E \cong S \oplus S^\perp$.

Proposition 2.7. *Let $S \subset E$ be a holomorphic sub-bundle of a holomorphic Hermitian vector bundle (E, h) over M. Let $\nabla^Q = \nabla^E|_{\mathcal{A}^0(S)} - \nabla^S : \mathcal{A}^0(S) \to \mathcal{A}^1(S^\perp) \cong \mathcal{A}^1(Q)$ where Q is the quotient bundle E/S then*

(1) *$\nabla^{S,Q}(f\sigma) = f\nabla^Q(\sigma)$ for all $\sigma \in H^0(U, S)$ and for all smooth function f defined on an open set U in M,*

(2) *$\langle \nabla^{S,Q}(s), t\rangle_h = 0$ for all $\sigma, \tau \in H^0(U, S)$,*

(3) *$\nabla^{S,Q}$ is of type $(1, 0)$.*

Proof. Part (1) follows by a straightforward computation,

$$\nabla^{S,Q}(f\sigma) = \nabla^E|_{\mathcal{A}^0(S)}(f\sigma) - \nabla^S(f\sigma) = df \otimes \sigma + f\nabla^E(\sigma) - df \otimes \sigma - f\nabla^S\sigma = f\nabla^Q(\sigma).$$

For (2) we observe that for any point $p \in M$ there is a local holomorphic frame $e_1, ..., e_r, (r = \text{rank } E)$ such that $e_1, ..., e_s$ is a local holomorphic frame of S which is unitary at p. Choose local holomorphic coordinates (U, z) near p so that p is the origin. Define $g = I + A$ where I_r is the $(r \times r)$-identity matrix and

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

with

$$A_1 = (-\frac{\partial < e_i, e_j >_h}{\partial z_k}(0))_{1\le i,j\le s};\;\; A_2 = (-\frac{\partial < e_i, e_j >_h}{\partial z_k}(0))_{s+1\le i,j\le r}.$$

Define a new frame of E by $e^{'} = ge, i.e., e_i^{'} = \sum_{j=1}^{r} g_{ij}e_j, i = 1, ..., r$. Then

(i) $e'_1, ..., e'_r$ is a holomorphic frame of E such that $e'_1, ..., e'_s$ is a holomorphic frame of S,

(ii) with respect to the frame e' the metric h is the identity at p,

(iii) with respect to the frame e',

$$dh = \begin{pmatrix} 0_s & * \\ * & 0_{r-s} \end{pmatrix}$$

at p where 0_μ denote the 0-matrix of dimension $\mu \times \mu$.

Let $\theta^E = (dh)h^{-1} = (\theta_{ij})_{1\le i,j\le r}$ be the connection matrix of h and $\theta^S = (dh_1)h_1^{-1} = (\tilde{\theta}_{ij})_{1\le i,j\le s}, h_1 = h|_S$ with respect to the frame $e^{'}$ then

$$\nabla^{S,Q}(e'_i) = \sum_{j=1}^{r} \theta_{ij}\, e_j^{'} - \sum_{j=1}^{s} \tilde{\theta}_{ij}\, e'_j.$$

At the point p, we have,

$$(dh)h^{-1} = \begin{pmatrix} 0_s & * \\ * & 0_{r-s} \end{pmatrix} \begin{pmatrix} I_s & 0 \\ 0 & I_{r-s} \end{pmatrix} = 0$$

hence we have $\theta_{ij} = 0$ for $1 \le i, j \le s$ and also for $s + 1 \le i, j \le r$. In particular we see that $\theta^S = 0$ at p. Thus, at the point p,

$$\nabla^{S,Q}(e'_i) = \sum_{j=s+1}^{r} \theta_i^j\, e'_j$$

and so $\langle \nabla^{S,Q}(e'_i), e'_j \rangle = 0$ for $1 \le i, j \le s$. Since p is an arbitrary point of M this proves (2). It remains to prove (3). Note that the property $\nabla^{S,Q}(fs) = f\nabla^{S,Q}(s)$ for smooth functions f implies that the type of $\nabla^{S,Q}$ is independent of the choice of admissible (i.e., $e_1, ..., e_s, e_{s+1}, ..., e_r$ is a local frame for E and $e_1, ..., e_s$ is a local frame for S) local frames. From the preceding we see that relative to an admissible frame, the matrix of 1-forms associated to $\nabla^{S,Q}$ is given by $B_{e^{'}} = (\theta_i^j)_{1\le i\le s, s=1\le j\le r}$ and if $\tilde{e}$ is another admissible frame then

$$\tilde{e}_l = \sum_{i=1}^{s} g_l^i e_i,\; 1 \le l \le s$$

thus

$$\nabla^{S,Q}\tilde{e}_l = \sum_{i=1}^{s} g_l^i \nabla^{S,Q} e_i = \sum_{i=1}^{s}\sum_{j=s+1}^{r} g_l^i \theta_i^j e_j.$$

It is clear from this that the type of the 1-forms is unchanged. If the frame is holomorphic then the forms are of type (1, 0) by Proposition 2.2. □

By the preceding proposition, with respect to a local unitary frame $e_1, ..., e_r$, ($r =$ rank E) of E such that $e_1, ..., e_s$ is a local unitary frame of S and $e_{s+1}, ..., e_r$ is a local unitary frame of $S^\perp = Q$,

$$\nabla^{S,Q} = \sum_{s+1\le\alpha\le r}\sum_{1\le\beta\le s} \theta^\alpha_\beta\, e_\alpha \otimes e^{*\beta},\ B = (\theta^\alpha_\beta)_{s+1\le\alpha\le r;1\le\beta\le s} \tag{2.14}$$

and B is of type (1, 0). By the skew Hermitian property, the matrix representing ∇^E is given by:

$$\theta_E = \begin{pmatrix} \theta_S & \overline{B}^t \\ B & \theta_{S^\perp}\end{pmatrix} = \begin{pmatrix} \theta_S & \overline{B}^t \\ B & \theta_Q\end{pmatrix},\ \theta_S = -\overline{\theta}^t_S,\ \theta_Q = -\overline{\theta}^t_Q$$

where $\overline{B}^t$ is the conjugate of the transpose of the matrix B. The curvature matrices are denoted by $\Theta_E = d\theta_E - \theta_E \wedge \theta_E, \Theta_S = d\theta_S - \theta_S \wedge \theta_S, \Theta_Q = d\theta_Q - \theta_Q \wedge \theta_Q$ respectively. By definition,

$$\Theta_E = \begin{pmatrix} d\theta_S - \theta_S\wedge\theta_S - \overline{B}^t\wedge B & d\overline{B}^t - \theta_S\wedge\overline{B}^t - \overline{B}^t\wedge\theta_Q \\ dB - B\wedge\theta_S - \theta_Q\wedge B & d\theta_Q - \theta_Q\wedge\theta_Q - B\wedge\overline{B}^t\end{pmatrix}$$

hence we have:

$$\Theta_E|_{\mathcal{A}^0(S)} = \Theta_S - \overline{B}^t\wedge B \ \text{ and } \ \Theta_E|_{\mathcal{A}^0(Q)} = \Theta_Q - B\wedge\overline{B}^t. \tag{2.15}$$

By (2.14) $\overline{B}^t \wedge B$ is an $(s\times s)$-matrix while $B\wedge\overline{B}^t$ is an $((r-s)\times(r-s))$-matrix of forms of type (1, 1):

$$\overline{B}^t\wedge B = \sum_{1\le\alpha,\beta\le s}\sum_{\gamma=s+1}^{r}(\overline{\theta}^\alpha_\gamma\wedge\theta^\gamma_\beta)\, e_\alpha\otimes e^{*\beta},$$

$$B\wedge\overline{B}^t = \sum_{s+1\le\alpha,\beta\le r}\sum_{\gamma=1}^{s}(\theta^\alpha_\gamma\wedge\overline{\theta}^\gamma_\beta)\, e_\alpha\otimes e^{*\beta}.$$

These matrices are skew Hermitian, hence $\sqrt{-1}B\wedge\overline{B}^t$ is a matrix of real (1, 1)-forms which shall be referred to as the *second fundamental form of S in E*. Given $\sigma = \sum_{\alpha=1}^s s^\alpha e_\alpha \in H^0(U,S)\setminus\{0\}$ and $v = \sum_{\alpha=1}^n v^\alpha \frac{\partial}{\partial z^\alpha} \in H^0(U, T^{1,0}M)\setminus\{0\}$ we have (keep in mind that $e_1, ..., e_r$ is unitary),

$$\begin{aligned}\langle\sqrt{-1}B\wedge\overline{B}^t(v,\overline{v})\sigma,\sigma\rangle_h &= \langle \sum_{s+1\le\alpha,\beta\le r}\sum_{\gamma=1}^{s}\theta^\alpha_\gamma(v)\overline{\theta}^\gamma_\beta(\overline{v})s^\beta e_\alpha, \sum_\eta s^\eta e_\eta\rangle_h \\ &= \sum_{s+1\le\alpha,\beta\le r}\sum_{\gamma=1}^{s}\theta^\alpha_\gamma(v)\overline{\theta}^\gamma_\beta(\overline{v})s^\beta\overline{s}^\alpha \\ &= \sum_{\gamma=1}^{s}(\sum_{s+1\le\alpha\le r}\theta^\alpha_\gamma(v)\overline{s}^\alpha \sum_{s+1\le\beta\le r}\overline{\theta}^\gamma_\beta(\overline{v})s^\beta) \\ &= \sum_{\gamma=1}^{s}|\sum_{s+1\le\alpha\le r}\theta^\alpha_\gamma(v)\overline{s}^\alpha|^2 \ge 0.\end{aligned}$$

This together with (2.15) yields,

Theorem 2.1. *Let $S \subset E$ be a holomorphic sub-bundle of a holomorphic hermitian vector bundle (E, h) over M. Then the second fundamental form of S in E is non-negative; consequently, we have*

$$\langle \frac{\sqrt{-1}}{2\pi}\Theta_E(v,\overline{v})s, s\rangle_h \;\geq\; \langle \frac{\sqrt{-1}}{2\pi}\Theta_S(v,\overline{v})\sigma, \sigma\rangle_h$$

for all $\sigma \in H^0(U, S) \setminus \{0\}$ and any $v \in H^0(U, T^{1,0}M) \setminus \{0\}$ of type $(1, 0)$ and

$$\langle \frac{\sqrt{-1}}{2\pi}\Theta_E(v,\overline{v})\tau, \tau\rangle_h \;\leq < \frac{\sqrt{-1}}{2\pi}\Theta_Q(v,\overline{v})\tau, \tau >_h$$

for all $\tau \in H^0(U, Q) \setminus \{0\}$ and any $v \in H^0(U, T^{1,0}M) \setminus \{0\}$ of type $(1, 0)$.

Recall that the curvature Θ is a section of $T^{*1,1}M \otimes \mathrm{Hom}(E, E)$. Thus for local vector fields u, v of type $(1, 0)$, $\Theta(u, \bar{v})$ is a section of $\mathrm{Hom}(E, E)$. Thus for a local section v of E, $\Theta(u, \bar{v})\sigma$ is a section of E.

Consider the special case $E = T^{1,0}M$, the *holomorphic* tangent bundle with Hermitian metric g. Then for non-zero vectors $u, v \in T^{1,0}M$ the *holomorphic bisectional curvature* of g, denoted $K(u, v)$, is defined as (see [], [])

$$K(u, v) = \frac{1}{2}\mathrm{Re}\frac{\sqrt{-1}}{2\pi}\frac{\langle\Theta(u,\overline{v})v, u\rangle_g}{||u||^2||v||^2}. \tag{2.16}$$

If $u = v$ then $K(u) = K(u, u)$ is the *holomorphic sectional curvature* of the Hermitian metric g. It is well-known that

$$\langle\Theta(u,\overline{v})v, u\rangle_g \;=\; \langle\Theta(u,\overline{u})v, v\rangle_g$$

if g is *Kähler*. Thus for a Kähler metric the holomorphic bisectional curvature $K(u, v)$ may also be defined as:

$$K(s, t) = \frac{1}{2}\mathrm{Re}\frac{\sqrt{-1}}{2\pi}\frac{\langle\Theta(u,\overline{u})v, v\rangle_g}{||u||^2||v||^2}. \tag{2.17}$$

For this reason we shall referred to, for any Hermitian holomorphic bundle (E, h) over a Kähler manifold (M, g)

$$K_E(u, \sigma) = \frac{1}{2}\mathrm{Re}\frac{\sqrt{-1}}{2\pi}\frac{\langle\Theta_E(u,\overline{u})\sigma, \sigma\rangle_h}{||u||_g^2||\sigma||_h^2} \tag{2.18}$$

as the holomorphic bisectional curvature at u in the direction of σ.

Corollary 2.1. *Let N be a complex submanifold of a Kähler manifold (M, g), equipped with the induced metric. Then the holomorphic bisectional curvatures satisfy the estimate $K_N(s, t) \leq K_M(u, v)$ for all non-zero vectors $u, v \in T^{1,0}N$.*

We do not need the Kähler condition for the holomorphic sectional curvature:

Corollary 2.2. *Let N be a complex submanifold of a Hermitian manifold (M, g), equipped with the induced metric. Then the holomorphic sectional curvatures satisfy the estimate $K_N(u) \leq K_M(u)$ for all non-zero vectors $u \in T^{1,0}N$.*

The first Chern form of (E, h) (resp. $(S, h|_S)$ and $(Q, (h|_S)^\perp$, we shall write for simplicity (S, h) and (Q, h)) is by definition the trace of $\frac{\sqrt{-1}}{2\pi}\Theta_E$ (resp. $\frac{\sqrt{-1}}{2\pi}\Theta_S$ and $\frac{\sqrt{-1}}{2\pi}\Theta_Q$), thus with respect to a unitary frame,

$$c_1(E, h) = \frac{\sqrt{-1}}{2\pi}\sum_{\alpha=1}^{r}(\Theta_E)^\alpha_\alpha$$

(resp., $c_1(S, h) = \frac{\sqrt{-1}}{2\pi}\sum_{\alpha=1}^{s}(\Theta_S)^\alpha_\alpha$, $c_1(Q, h) = \frac{\sqrt{-1}}{2\pi}\sum_{\alpha=s+1}^{r}(\Theta_Q)^\alpha_\alpha$). By (2.12),

$$\begin{aligned}
c_1(S, h) &= \frac{\sqrt{-1}}{2\pi}(\sum_{\alpha=1}^{s}(\Theta_E)^\alpha_\alpha + \sum_{1\leq\alpha\leq s}\sum_{\gamma=s+1}^{r}\overline{\sigma}^\alpha_\gamma \wedge \sigma^\gamma_\alpha) \\
&= \frac{\sqrt{-1}}{2\pi}(\sum_{\alpha=1}^{s}(\Theta_E)^\alpha_\alpha - \sum_{s+1\leq j,l\leq r}\sum_{k=1}^{s}\sigma^\alpha_\gamma \wedge \overline{\sigma}^\gamma_\alpha) \\
c_1(Q, h) &= \frac{\sqrt{-1}}{2\pi}(\sum_{\alpha=1}^{s}(\Theta_E)^\alpha_\alpha + \sum_{s+1\leq j,l\leq r}\sum_{k=1}^{s}\sigma^\alpha_\gamma \wedge \overline{\sigma}^\gamma_\alpha).
\end{aligned}$$

With these we have:

Corollary 2.3. *Let (E, h) be a Hermitian holomorphic vector bundle over a complex manifold M and S be a holomorphic sub-bundle of E with quotient bundle $Q = E/S$. If $c_1(E, h)$ is positive semi-definite then $c_1(Q, h)$ is also positive semi-definite. If $c_1(E, h)$ is negative semi-definite then $c_1(S, h)$ is negative semi-definite. If $c_1(E, h) = 0$ then $c_1(S, h) \leq 0$ and $c_1(Q, h) \geq 0$.*

Proof. The condition $c_1(E, h)|_{\mathcal{A}^0(Q)} \geq 0$ implies that $\frac{\sqrt{-1}}{2\pi}\sum_{\alpha=s+1}^{r}(\Theta_E)^\alpha_\alpha \geq 0$ hence $c_1(Q, h) \geq 0$ because the second fundamental form is also non-negative. On the other hand, if $c_1(E, h)$ is negative semi-definite then $c_1(E, h)|_{\mathcal{A}^0(S)} \leq 0$ which means that $\frac{\sqrt{-1}}{2\pi}\sum_{\alpha=1}^{s}(\Theta_E)^\alpha_\alpha \leq 0$ hence $c_1(S, h) \leq 0$. QED

Department of Mathematics, University of Notre Dame, Notre Dame, Indiana 46556 USA; e-mail: pmwong@nd.edu