Nevanlinna Theory

0.1 The First Main Theorem

We begin by recalling the following well-known Poisson-Jensen formula in the classical complex analysis.

Theorem A1.1.1 (Poisson-Jensen Formula) Let $f \not\equiv 0$ be meromorphic on the closed disc $\overline{\mathbf{D}}(R)$, $R < \infty$. Let $a_1, \ldots, a_p$ denote the zeros of f in $\overline{\mathbf{D}}(R)$, counting multiplicities, and let $b_1, \ldots, b_q$ denote the poles of f in $\overline{\mathbf{D}}(R)$, also counting multiplicities. Then for any z in $|z| < R$ which is not a zero or pole, we have

$$\begin{aligned} \log|f(z)| &= \int_0^{2\pi} \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} \log|f(Re^{i\theta})| \frac{d\theta}{2\pi} \\ &\quad - \sum_{i=1}^{p} \log\left| \frac{R^2 - \bar{a}_i z}{R(z - a_i)} \right| + \sum_{j=1}^{q} \log\left| \frac{R^2 - \bar{b}_j z}{R(z - b_j)} \right|. \end{aligned}$$

Proof. We note that it suffices to prove the theorem when f has no zeros or poles on the circle $|z| = R$. Otherwise, we consider the function $f(\rho z)$ and let $\rho \to 1$.

We first consider the case when f is analytic and has no zeros in the closed disc $|z| \le R$. Then $\log|f|$ is harmonic. For a given z in $\mathbf{D}(R)$, we consider the linear transformation $L(w) = \dfrac{R^2(z - w)}{R^2 - \bar{z}w}$. L sends z to zero and satisfies $|L(w)| = R$ if $|w| = R$. Let $F(w) = \log f(L(w))$. Applying the Mean Value Theorem for harmonic functions to $F(w)$, we have

$$\log f(z) = F(0) = \int_0^{2\pi} F(Re^{i\theta}) \frac{d\theta}{2\pi} = \int_{|w|=R} F(w) \frac{dw}{2\pi i w}. \tag{1}$$

We let $\zeta = L(w)$, then

$$w = L^{-1}(\zeta) = \frac{R^2(z - \zeta)}{R^2 - \bar{z}\zeta}.$$

So, for $|\zeta| = R$,

$$\begin{aligned} \frac{dw}{2\pi i w} &= \frac{1}{2\pi i}\left(\frac{-1}{z - \zeta} + \frac{\bar{z}}{R^2 - \bar{z}\zeta} \right) d\zeta = \left(\frac{-1}{z - \zeta} + \frac{\bar{z}}{\bar{\zeta}\zeta - \bar{z}\zeta} \right) \frac{d\zeta}{2\pi i} \\ &= \left(\frac{-\zeta}{z - \zeta} + \frac{\bar{z}}{\bar{\zeta} - \bar{z}} \right) \frac{d\zeta}{2\pi i\zeta} = \frac{R^2 - |z|^2}{|\zeta - z|^2} \frac{d\zeta}{2\pi i \zeta}. \end{aligned} \tag{2}$$

Note that when $|w| = R$, $|\zeta| = R$, and $\frac{d\zeta}{i\zeta} = d\theta$, so by combining (1) and (2)

$$\log f(z) = \int_0^{2\pi} \log f(Re^{i\theta}) \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} \frac{d\theta}{2\pi}.$$

Thus

$$\log |f(z)| = \int_0^{2\pi} \log |f(Re^{i\theta})| \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} \frac{d\theta}{2\pi}. \tag{3}$$

The Theorem is proved in this case.

For the general case, we consider the function

$$g(z) = f(z) \frac{\prod_{\mu=1}^{p} \frac{R^2 - \bar{a}_\mu z}{R(z - a_\mu)}}{\prod_{\nu=1}^{q} \frac{R^2 - \bar{b}_\nu z}{R(z - b_\nu)}}.$$

Then g has no zeros or poles in $|z| \le R$. Note that when $|z| = R$, $|g(z)| = |f(z)|$. Applying (1.3) to g yields the theorem.

Let $z_0 \in \mathbf{D}(R)$. If $f(z) = c(z - z_0)^m + \cdots$, where c is the leading nonzero coefficient, then m is called the order of f at z_0 and is denoted by $\mathrm{ord_{z_0}f}$.

Corollary A1.1.3 (Jensen's Formula) *Let $f \not\equiv 0$ be meromorphic on $\overline{\mathbf{D}}(R)$, $R < \infty$. Let $a_1, \ldots, a_p$ denote the zeros of f in $\overline{\mathbf{D}}(R) - \{0\}$, counting multiplicities, and let $b_1, \ldots, b_q$ denote the poles of f in $\overline{\mathbf{D}}(R) - \{0\}$, also counting multiplicities. Then*

$$\log |c_f| = \int_0^{2\pi} \log |f(Re^{i\theta})| \frac{d\theta}{2\pi} - \sum_{\mu=1}^{p} \log \left| \frac{R}{a_\mu} \right| + \sum_{\nu=1}^{q} \log \left| \frac{R}{b_\nu} \right| - (\mathrm{ord_0 f}) \log \mathrm{R},$$

where $f(z) = c_f z^{\mathrm{ord_0 f}} + \cdots$, $\mathrm{ord_0 f} \in \mathbf{Z}$, and c_f is the leading nonzero coefficient.

Proof. Applying Theorem A1.1.1 with $z = 0$ to the function

$$f(z) z^{-\mathrm{ord_0 f}}.$$

We now proceed to define Nevanlinna functions. Let f be a meromorphic function on $\mathbf{D}(R)$, where $0 < R \le \infty$ and let $r < R$. Denote the number of poles of f on the closed disc $\overline{\mathbf{D}}(r)$ by $n_f(r,\infty)$, counting multiplicity. We then define the **counting function** $N_f(r,\infty)$ to be

$$N_f(r,\infty) = n_f(0,\infty)\log r + \int_0^r [n_f(t,\infty) - n_f(0,\infty)]\frac{dt}{t},$$

here $n_f(0,\infty)$ is the multiplicity if f has a pole at $z=0$. For each complex number a, we then define the **counting function** $N_f(r,a)$ to be

$$N_f(r,a) = N_{1/(f-a)}(r,\infty). \tag{4}$$

So, in particular, by the definition of the Lebesgue-Stieltjes integral,

$$N_f(r,0) = (\mathrm{ord}_0^+ \mathrm{f})\log \mathrm{r} + \sum_{\mathrm{z}\in\mathbf{D}(\mathrm{r}),\mathrm{z}\neq 0} (\mathrm{ord}_\mathrm{z}^+ \mathrm{f})\log\left|\frac{\mathrm{r}}{\mathrm{z}}\right| \tag{5}$$

where $\mathrm{ord}_\mathrm{z}^+\mathrm{f} = \max\{0, \mathrm{ord}_\mathrm{z}\mathrm{f}\}$ is just the multiplicity of the zero at z. We note that $N_f(r,a)$ measures how many times f takes value a. With this notation, we can rewrite Corollary A1.1.3 as

Corollary A1.1.4 *Let $f \not\equiv 0$ be meromorphic on $\overline{\mathbf{D}}(r)$. Then*

$$\log|c_f| = \int_0^{2\pi} \log|f(re^{i\theta})|\frac{d\theta}{2\pi} - \sum_{z\in\mathbf{D}(r),z\neq 0} (\mathrm{ord}_\mathrm{z}\mathrm{f})\log\left|\frac{\mathrm{r}}{\mathrm{z}}\right| - (\mathrm{ord}_0\mathrm{f})\log\mathrm{r},$$

or equivalently,

$$\log|c_f| = \int_0^{2\pi} \log|f(re^{i\theta})|\frac{d\theta}{2\pi} + N_f(r,\infty) - N_f(r,0).$$

The Nevanlinna's **proximity function** $m_f(r,\infty)$ is defined by

$$m_f(r,\infty) = \int_0^{2\pi} \log^+|f(re^{i\theta})|\frac{d\theta}{2\pi}, \tag{6}$$

where $\log^+ x = \max\{0, \log x\}$. For any complex number a, the **proximity function** $m_f(r,a)$ of f with respect to a is then defined by

$$m_f(r,a) = m_{1/(f-a)}(r,\infty). \tag{7}$$

We note that $m_f(r,a)$ measures how close f is, on average, to a on the circle of radius r. Finally, the **Nevanlinna's characteristic function** of f is defined by

$$T_f(r) = m_f(r,\infty) + N_f(r,\infty). \tag{8}$$

$T_f(r)$ measures the growth of f. For example: $T_f(r) = O(1)$ if and only if f is constant; $T_f(r) = O(\log r)$ if and only if f is a rational function.

The characteristic function T, the proximity function m and the counting function N are the three main **Nevanlinna functions**. Nevanlinna theory can be described as the study of how the growth of these three functions is interrelated. The First Main Theorem is a reformulation of Corollary A1.1.4.

Theorem A1.1.5 (First Main Theorem) *Let $f \not\equiv 0$ be meromorphic on $\overline{\mathbf{D}}(R)$, $R \leq \infty$. Then, for any $0 \leq r < R$,*

(i) $T_f(r) = m_f(r,0) + N_f(r,0) + \log|c_f|$.

(ii) Given a complex number a,

$$|T_f(r) - m_f(r,a) - N_f(r,a)| \leq \left|\log|c_{1/(f-a)}|\right| + \log^+|a| + \log 2,$$

where $c_{1/(f-a)}$ is the leading non-zero coefficient in the Taylor's expansion of $1/(f-a)$ around 0.

Proof. (i) is derived directly from Corollary A1.1.4. To prove (ii), applying Corollary A1.1.4 to $1/(f-a)$ yields

$$\log|c_{1/(f-a)}| = \int_0^{2\pi} \log\frac{1}{|f(re^{i\theta}) - a|}\frac{d\theta}{2\pi} + N_{1/(f-a)}(r,\infty) - N_{1/(f-a)}(r,0).$$

Since $\log x = \log^+ x - \log^+(1/x)$,

$$\begin{aligned}\log|c_{1/(f-a)}| &= \int_0^{2\pi} \log^+\frac{1}{|f(re^{i\theta}) - a|}\frac{d\theta}{2\pi} - \int_0^{2\pi}\log^+|f(re^{i\theta}) - a|\frac{d\theta}{2\pi} \\ &\quad + N_f(r,a) - N_f(r,\infty).\end{aligned}$$

Thus,

$$\int_0^{2\pi} \log^+|f(re^{i\theta}) - a|\frac{d\theta}{2\pi} = -N_f(r,\infty) + m_f(r,a) + N_f(r,a) - \log|c_{1/(f-a)}|.$$

Note that if x and y are positive real numbers, then

$$\log^+(x+y) \leq \log^+ 2\max\{x,y\} \leq \log^+ x + \log^+ y + \log 2.$$

So

$$|\log^+|x-y| - \log^+|x|| \leq \log^+|y| + \log 2.$$

Thus

$$|T_f(r) - m_f(r,a) - N_f(r,a) + \log|c_{1/(f-a)}|| \leq \log^+|a| + \log 2.$$

Lemma A1.1.6 *For any $a \in \mathbf{C}$,*

$$\int_0^{2\pi} \log|a - e^{i\theta}| \frac{d\theta}{2\pi} = \log^+|a|.$$

Proof. If $|a| > 1$, then the function $z - a$ has no zeros in $|z| < 1$, and $\log^+|a| = \log|a|$, so the formula holds by Jensen's formula. If $|a| < 1$, then, by Jensen's formula

$$\log|a| = \int_0^{2\pi} \log|a - e^{i\theta}| \frac{d\theta}{2\pi} + \log|a|,$$

so $\int_0^{2\pi} \log|a - e^{i\theta}| \frac{d\theta}{2\pi} = 0$. This proves the lemma.

Let, for $|z| < R$,

$$G_a(z) = \frac{R^2 - \bar{a}z}{R(z-a)}.$$

Lemma A.1.1.7 For $r < R$,

$$m_{G_a}(r) := \int_0^{2\pi} \log^+|G_a(re^{i\theta})| \frac{d\theta}{2\pi} = \log\frac{R}{r} - \log^+|\frac{a}{r}|.$$

Proof. Notice that, for $|z| \leq R$, $\log^+|G_a(re^{i\theta})| = \log|G_a(re^{i\theta})|$. This immediately follows from the lemma above.

Proposition *Let $G = G_{R,f}^{\infty} =: \prod_{|a|<R, f(a)=\infty} \frac{R^2-\bar{a}z}{R(z-a)}$. Then*

$$m_G(r) = N_f(R,\infty) - N_f(r,\infty),$$

and similarly with 0 replacing ∞.

0.2 The Logarithmic Derivative Lemma

In this section, we derive the Logarithmic Derivative Lemma. Let f or h be meromorphic function on $|z| \leq R$ with R fixed until otherwise specified. We let

$$r < s < R$$

and let $|z| = r$.

We first study f'/f when f has no zeros and poles.

Lemma A1.2.1 *Suppose h is holomorphic without zeros on $|z| \leq s$. Then*

$$m_{h'/h}(r) \leq \log^+ s + 2\log^+ \frac{1}{s-r} + \log^+ \max[m_h(s), m_{1/h}(s)] + 2\log 2.$$

Proof. Since

$$\log|h(z)| = \frac{1}{2}[\log h(z) + \log \bar{h}(z)],$$

and $(\log \bar{h}(z))' = 0$,

$$\frac{h'(z)}{h(z)} = (\log h(z))' = 2(\log|h(z)|)'.$$

From Jensen's formula

$$\log|h(z)| = \int_0^{2\pi} \log|h(se^{i\theta})| \frac{s^2 - |z|^2}{|se^{i\theta} - z|^2} \frac{d\theta}{2\pi}.$$

Differentiating with respect to z under the integral to get

$$h'/h(z) = \int_0^{2\pi} \log|h(se^{i\theta})| \frac{2se^{i\theta}}{(se^{i\theta} - z)^2} \frac{d\theta}{2\pi}.$$

Using $|\log\alpha| = \log^+ \alpha + \log^+ 1/\alpha$. Then

$$|h'/h(z)| \leq \frac{2s}{(s-r)^2}[m_h(s) + m_{1/h}(s)].$$

Taking $\log^+$ and integrating proves the lemma.

Next we deal with the **canonical product**. Let

$$G_s^0(z) = \prod_{f(a)=0} \frac{s^2 - \bar{a}z}{s(z-a)}, \qquad G_s^\infty(z) = \prod_{f(a)=\infty} \frac{s^2 - \bar{a}z}{s(z-a)},$$

where the products are taken with the multiplicities. Let

$$P = P_s = G_s^\infty(z)/G_s^0(z), \qquad G = G_s = G_s^0(z)G_s^\infty(z).$$

Then

$$h = fP^{-1}$$

has no zeros and poles in $|z| < s$ and $f = hP$. We are able to estimate $m_{f'/f}$ in terms of $m_{P'/P}$ and $m_{G'/G}$.

Recall that each term in the product of $G_s^0(z)$ and $G_s^\infty(z)$ has absolute values ≥ 1, so, for $|z| < s$,

$$\log |G_s^0(z)| = \log^+ |G_s^0(z)|, \qquad \log |G_s^\infty(z)| = \log^+ |G_s^\infty(z)|.$$

Hence $|P| \leq |G|$ and $|1/P| \leq |G|$. We then obtain for $r \geq 1$ or all $r \geq 0$ if $f(0) \neq 0, \infty$:

$$m_h \leq m_f + m_{1/P} \leq m_f + m_G \leq T_f + m_G,$$

$$m_{1/h} \leq m_{1/f} + m_{1/P} \leq m_{1/f} + m_G \leq T_f + m_G,$$

since we assume that $c_f = 1$, so $T_{1/f} = T_f$. Since $m_G(s) = 0$, we can re-write the above lemma as

Lemma A1.2.2 *Assume that $c_f = 1$. Then for $r \geq 1$ or all $r \geq 0$ if $f(0) \neq 0, \infty$,*

$$m_{h'/h}(r) \leq \log^+ R + 2\log^+ \frac{1}{s-r} + \log^+ T_f(R) + 2\log 2.$$

Next we give a bound of $m_{G'/G}$ and $m_{P'/P}$.

Lemma A1.2.3 Let $n_f(s, 0+\infty) = n_f(s,0) + n_f(s,\infty)$. Then, for $0 < r < s$ we have

$$m_{P'/P} \leq \log^+ \frac{s}{(s-r)^2} + \frac{s-r}{R-s}\frac{R}{r}N_f(R, 0+\infty) + \log^+ \left[\frac{R}{R-s}N_f(R, 0+\infty)\right].$$

Proof. Consider one multiplication term

$$G_a(z) = \frac{s^2 - \bar{a}z}{s(z-a)}.$$

Then

$$-G_a'/G_a(z) = \frac{s^2 - |a|^2}{(z-a)(s^2 - \bar{a}z)}.$$

But

$$|s^2 - \bar{a}z| \geq s^2 - \bar{a}r \geq s(s-r).$$

Therefore

$$|G_a'/G_a(z)| \leq \frac{s}{(s-r)^2}|G_a(z)|.$$

We use the fact that $|G_a(z)| \geq 1$ and the fact that $Q \mapsto Q'/Q$ is a homomorphism. Then $|G'/G|$ and $|P'/P| \leq \sum_a |G_a'/G_a| \leq \frac{s}{(s-r)^2}|G_a|$. The sum is over zeros and poles. Apply $\log^+$and integrate, we get, by Proposition in the last section,

$$m_{P'/P} \leq \log^+ \frac{s}{(s-r)^2} + \sum_a m_{G_a}(r) + \log^+ n_f(s, 0+\infty)$$

$$\leq \log^+ \frac{s}{(s-r)^2} + N_f(s, 0+\infty) - N_f(r, 0+\infty) + \log^+ n_f(s, 0+\infty).$$

We now have to estimate N_f and n_f. We prove the following lemma which will conclude the proof of Lemma A1.2.3.

Lemma A1.2.4 Let $n(r)$ be a monotone increasing function of r for $0 \leq r \leq R$, and let

$$N(r) = \int_0^r [n(t) - n(0)]\frac{dt}{d} + n(0)\log r.$$

Let $0 < s < R$. If $s \geq 1$ or $n(0) = 0$, then

$$n(s) \leq \frac{R}{R-s}(N(R) - N(s)) \leq \frac{R}{R-s}N(s),$$

and similarly for $r < s$,

$$N(s) - N(r) \leq n(s)\frac{s-r}{r} \leq \frac{s-r}{R-s}\frac{R}{r}N(R).$$

Proof. We shall use

$$\frac{R-s}{R} \leq \log\frac{R}{s} \leq \frac{R-s}{s}.$$

Then

$$n(s) = \frac{1}{\log\frac{R}{s}} n(s) \int_s^R \frac{dt}{t} \leq \frac{1}{\log\frac{R}{s}} \int_s^R \frac{n(t)}{t} dt \leq \frac{1}{\log\frac{R}{s}} (N(R) - N(s)).$$

$$N(s) - N(r) = \int_r^s n(t)\frac{dt}{t} \leq n(s)\log(s/r) \leq n(s)\frac{s-r}{r}.$$

We now put the lemmas together. We start with

$$m_{f'/f} \leq m_{h'/h} + m_{P'/P} + \log 2$$

and obtain

$$\begin{aligned} m_{f'/f}(r) \;\leq\;& 3\log^+ R + 4\log^+\frac{1}{s-r} + \log^+\frac{1}{R-s} + 3\log^+ T(R) + 4\log 2 + \\ &+ \frac{s-r}{R-s}\frac{R}{r} N_f(R, 0+\infty). \end{aligned}$$

The last term is obviously the worst, so we make it small and fix it up so that the other terms will be founded as desired. Namely given $r < R$ we choose s such that

$$\frac{s-r}{R-s}\frac{R}{r} = \frac{\frac{1}{2}}{T+1},$$

where $T = T_f(R)$. Assume that $r \geq 1$ to ensure $T \geq 0$. Then the last term satisfies

$$\frac{s-r}{R-s}\frac{R}{r} N_f(R, 0+\infty) \leq 1.$$

From our choice of s, it then folows at once that s is to the left of the midpoint between r and R. Theorefore

$$\frac{1}{s-r} = \frac{R}{r}\frac{2(T+1)}{R-s} \leq \frac{R}{r}\frac{4}{R-r}(T+1)$$

and

$$\frac{1}{s-r} \leq \frac{2}{R-r}.$$

Thus we can get rid ourselves of s, and get an estimate entirely in terms of r and R, nemely:

Proposition 1.2.5

$$m_{f'/f}(r) \le 3\log^+ R + 4\log^+ \frac{R}{r} + 5\log^+ \frac{1}{R-r} + 7\log^+ T(R) + 17\log 2 + 1.$$

Lemma A1.2.4 (Borel's Growth Lemma) *Let $F(r)$ be a positive, non-decreasing, continuous function defined on $[r_0, \infty)$ with $r_0 \ge e$ such that $F(r) \ge e$ on $[r_0, \infty)$. Then, for every $\epsilon > 0$, there exists a closed set $E \subset [r_0, \infty)$ (called the "exceptional set") of finite Lebesgue measure such that if we set $\rho = r + 1/\log^{1+\epsilon} F(r)$ for all $r \ge r_0$ and not in E, we have*

$$\log F(\rho) \le \log F(r) + 1 \tag{9}$$

and

$$\log^+ \frac{\rho}{r(\rho - r)} \le (1+\epsilon)\log^+ \log F(r) + \log 2. \tag{10}$$

Proof Let

$$E = \left\{ r \in [r_0, \infty) : \ F\left(r + \frac{1}{\log^{1+\epsilon} F(r)} \right) \ge eF(r) \right\}.$$

We may assume that E is non-empty, otherwise, the lemma is trivial. We claim that E is of finite Lebesgue measure.

Let r_1 be the smallest $r \in E$ with $r \ge r_0$. Now assume that we have found numbers $r_1, \ldots, r_n, s_1, \ldots, s_{n-1}$. We describe here how to inductively extend this set, and we continue this process as long as possible. If there is no number s with $F(s) \ge eF(r_n)$, then we stop here. Otherwise, by continuity of F, there exists an s with $F(s) = eF(r_n)$. Let s_n be the smallest such s. Then, if there is an $r \in E$ with $r \ge s_n$, let r_{n+1} be the smallest such r. Otherwise, we stop here.

For each pair r_j, s_j, clearly $s_j > r_j$, and since $r_j \in E$,

$$F\left(r_j + \frac{1}{\log^{1+\epsilon} F(r_j)} \right) \ge eF(r_j) = F(s_j).$$

Since F is nondecreasing, this implies

$$r_j + \frac{1}{\log^{1+\epsilon} F(r_j)} \geq s_j,$$

and so

$$s_j - r_j \leq \frac{1}{\log^{1+\epsilon} F(r_j)}. \tag{11}$$

Moreover, $F(r_{j+1}) \geq F(s_j) = eF(r_j)$ since $s_{j+1} \geq s_j$. Hence,

$$F(r_{n+1}) \geq eF(r_n) \geq e^2 F(r_{n-1}) \geq \cdots \geq e^n F(r_1) \geq e^{n+1}. \tag{12}$$

It follows that either we can only find finitely many r_n or else the sequence r_n goes to the infinity as n goes to the infinity. Since the set E is contained in the union of $[r_n, s_n]$, if we can only find finitely many r_n, then E is of finite Lebesgue measure. Now consider the case where n goes to ∞. Let $m(E)$ be the Lebesgue measure of E, then

$$m(E) \leq \sum_{n=1}^{\infty}(s_n - r_n).$$

By (11) and (12),

$$\sum_{n=1}^{\infty}(s_n - r_n) \leq \sum_{n=1}^{\infty} \frac{1}{\log^{1+\epsilon} F(r_n)} \leq \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}} < +\infty.$$

Thus the claim is proved.

To verify (9), let $r \geq r_0$ where r is not contained in E, then, by the construction of E,

$$F(\rho) = F\left(r + \frac{1}{\log^{1+\epsilon} F(r)}\right) \leq eF(r).$$

Thus $\log F(\rho) \leq \log F(r) + 1$. So (9) holds. Finally, we verify (10).

$$\frac{\rho}{r(\rho - r)} = \frac{1}{\rho - r} + \frac{1}{r} \leq \log^{1+\epsilon} F(r) + 1 \leq 2\log^{1+\epsilon} F(r).$$

Hence

$$\log^+ \frac{\rho}{r(\rho - r)} \leq (1+\epsilon)\log^+ \log F(r) + \log 2.$$

Theorem A1.2.5 (Lemma on the Logarithmic Derivative) *Let f be a non-constant meromorphic function on* $\mathbf{C}$. *Assume that $T_f(r_0) \geq e$ for some $r > 1$ and $c_f = 1$. Then, for every $\epsilon > 0$, the inequality*

$$m_{f'/f}(r, \infty) \leq 7 \log^+ T_f(r) + 4 \log r + 5(1+\epsilon) \log^+ \log^+ T_f(r) + 17 \log 2 + 5$$

holds for all $r \geq 1$ outside a set $E \subset (1, +\infty)$ with finite Lebesgue measure, where C is a constant which depends only on f.

Proof. Take $R = r + \frac{1}{\log^{1+\epsilon} T_f(r)}$. So, outside a set $E \subset (1, +\infty)$,

$$\log^+ T_f(R) \leq \log^+ T_f(r) + 1,$$

$$5 \log^+ \frac{1}{R - r} \leq (1+\epsilon) \log^+ \log^+ T_f(r).$$