Some Concepts and Results in Complex Geometry

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1 Geometry on Hermitian and Kahler Manifolds

• A complex manifold is a differentiable manifold admitting an open cover \( \{ U_\alpha \} \) and coordinate maps \( \phi_\alpha : U_\alpha \to \mathbb{C}^n \) such that \( \phi_\alpha \circ \phi_\beta^{-1} \) is holomorphic on \( \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^n \) for all \( \alpha, \beta \).

• A function \( f \) on an open set \( U \subset M \) is holomorphic if, for all \( \alpha \), \( f \circ \phi_\alpha^{-1} \) is holomorphic on \( \phi_\alpha(U_\alpha \cap U) \subset \mathbb{C}^n \). A map \( f : M \to N \) of complex manifolds is holomorphic if it is holomorphic given in terms of the local holomorphic coordinates of \( M \) and \( N \).

• Examples of complex manifolds include: 1. One-dimensional complex manifolds, which are called the Riemann surfaces; \( \mathbb{P}^n(\mathbb{C}) \), and complex torus \( \mathbb{C}/\Lambda \), where \( \Lambda \) is a discrete lattice.

• For \( p \in M \), let \( (z_1, \ldots, z_n) \) be a local holomorphic coordinates. Define

\[
\frac{\partial}{\partial z^i} = \frac{1}{2} \left( \frac{\partial}{\partial x^i} - \sqrt{-1} \frac{\partial}{\partial y^i} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}^i} = \frac{1}{2} \left( \frac{\partial}{\partial x^i} + \sqrt{-1} \frac{\partial}{\partial y^i} \right),
\]

\[
\partial = \sum \frac{\partial}{\partial z^i} \otimes dz^i, \quad \bar{\partial} = \sum \frac{\partial}{\partial \bar{z}^i} \otimes d\bar{z}^i, \quad \text{and} \quad d = \partial + \bar{\partial}.
\]

The complexified tangent space is

\[
T_{C,p}(M) =: \mathbb{C} \otimes T_p(M) = \left\{ \sum_{i=1}^n a_i \frac{\partial}{\partial x^i}|_p + \sum_{i=1}^n b_i \frac{\partial}{\partial y^i}|_p \mid a^i, b^i \in \mathbb{C} \right\}
\]
The holomorphic tangent space $T^1_p(M)$ and the antiholomorphic tangent space $T^0_p(M)$, for $p \in M$, are given by

$$T^1_p(M) = C \left\{ \frac{\partial}{\partial z^i} \bigg|_p \right\}_{i=1}^n, \quad T^0_p(M) = C \left\{ \frac{\partial}{\partial \bar{z}^j} \bigg|_p \right\}_{j=1}^n,$$

so that

$$T_{C,p}(M) = T^1_p(M) \oplus T^0_p(M).$$

$T^{(1,0)}(M) = \bigcup_{p \in M} T^{1,0}_p(M)$ is called the holomorphic tangent bundle. $\Gamma(M, T^{(1,0)}(M))$ is the set of smooth sections of $T^{(1,0)}(M)$, which is also called the smooth vector fields.

• A Hermitian metric on $M$, denoted by $ds^2$, is a set of Hermitian inner-product $\{\langle \cdot, \cdot \rangle_p\}_{p \in M}$ on $T^{(1,0)}_p(M)$ such that if $\xi, \eta$ are $C^\infty$ section of $T^{(1,0)}_p(M)$ over an open set $U$, then $\langle \xi, \eta \rangle$ is the $C^\infty$ function on $U$. If $z^1, \ldots, z^n$ is a local coordinate system of $M$, we write

$$ds^2 = \sum g_{i\bar{j}} dz^i \otimes d\bar{z}^j,$$

then, if $\xi = \xi^i \frac{\partial}{\partial z^i}$, $\eta = \eta^j \frac{\partial}{\partial \bar{z}^j}$,

$$\langle \xi, \eta \rangle = \sum g_{i\bar{j}} \xi^i \eta^j.$$

We also have $g_{i\bar{j}} = \overline{g_{\bar{j}i}}$. A complex manifold with a given Hermitian metric is said a Hermitian manifold.

• The metric form (associated to $ds^2$) is the form $(1, 1)$-form $\Phi = \sqrt{-1} 2 g_{i\bar{j}} dz^i \wedge d\bar{z}^j$. $(M, ds^2)$ is said to be Kahler if $d\Phi = 0$.

• The connection is a linear operator $D : \Gamma(M, T^{(1,0)}) \longrightarrow \Gamma(M, \Lambda^1(M) \otimes T^{(1,0)})$, where $\Lambda^1(M)$ is set of (smooth) differential forms of degree 1 on $U$ satisfies

$$(1.24) \quad D(fs) = df \otimes s + fDs,$$

for $\forall s \in \Gamma(M, T^{(1,0)})$ and $f$ is a smooth function on $M$. The connection gives a ”way” to differentiate the vector fields.
The connection $D$ is called the Hermitian connection if (i) $D$ is compatible with the complex structure, i.e. if we split $T_C^* = T^{(1,0)*} \oplus T^{(0,1)*}$ and write $D = D' + D''$, then $D'' = \partial$, (ii) $D$ is compatible with the Hermitian metric, $\langle d<\xi,\eta> = \langle D\xi,\eta > + \langle \xi,D\eta >$ where $\xi,\eta \in \Gamma(M,T^{(1,0)})$. Write $D' = dz^i \otimes \nabla_i$, then $\nabla_i$ is called the covariant derivative. $D$ also extends naturally to any type of tensor fields. From now on, the connection is always assumed to be the Hermitian connection.

In terms of local coordinate $(z^1, \ldots, z^n)$, write

$$D \frac{\partial}{\partial z^i} = \sum_{j=1}^{n} \omega^j_i \frac{\partial}{\partial z^j},$$

where $\omega = (\omega^j_i)$ is a $n \times n$ matrix whose entries are all 1-forms. $\omega$ is called the connection matrix.

For the Hermitian connection $D$, its connection matrix satisfies $\omega = \partial g \cdot g^{-1}$ (so the entries of $\omega$ are $(1,0)$-forms), or equivalently, $\omega^j_i = \frac{\partial g^{ij}}{\partial z^k} dz^k$.

Write

$$\omega^j_i = \sum_{k=1}^{n} \Gamma^j_{ik} dz^k$$

where the functions $\Gamma^j_{ik}$ are called the Christoffel symbols. From above, $\Gamma^j_{ik} = \frac{\partial g^{ij}}{\partial z^k} g^{kj}$.

$M$ is Kahler if and only if $\Gamma^k_{ji} = \Gamma^k_{ij}$. Th other equivalent conditions for being Kahler is (3) For $\forall p \in M$, there is a $C^\infty$ function $\phi$ on an open neighborhood of $p$, such that $\Phi = \sqrt{-1} \partial \bar{\partial} \phi$; (4) For $\forall p \in M$, there exists a local holomorphic coordinate system $z^1, \ldots, z^n$, such that $g_{ij}(p) = \delta_{ij}, dg_{ij}(p) = 0$. Such a coordinate is said to be normal at $p$. 


• For $\xi \in \Gamma(M, T^{(1,0)})$, in terms of local coordinate $(z^1, \ldots, z^n)$, write $\xi = \sum_{j=1}^n \xi^j \frac{\partial}{\partial z^j}$. Then

$$D\xi = \sum_{j=1}^n d\xi^j \frac{\partial}{\partial z^j} + \sum_{i,j=1}^n \xi^j \omega^i_j \frac{\partial}{\partial z^i},$$

or

$$\nabla_i \xi^j = \frac{\partial \xi^j}{\partial z^i} + \sum_{k=1}^n \Gamma^j_{ik} \xi^k.$$

Note that, for covariant tensor field $\{\xi^j\}$, the resulting $\{\nabla_i \xi^j\}$ (when $i$ is fixed) is still a covariant tensor field.

• The connection also extends to $\Gamma(M, \Lambda^k(M) \otimes T^{(1,0)})$ (where $\Lambda^k(M)$ is the set of k-forms): $D(\omega \otimes \xi) = d\omega \otimes \xi + (-1)^k \omega \wedge D\xi$. In particular, if, for $\omega = \sum_{j=1}^n \phi^j dz^j$ (contra-variant tensor field), then

$$\nabla_i \omega = \left( \frac{\partial \phi_j}{\partial z^i} - \sum_{k=1}^n \Gamma^k_{ij} \phi_k \right) dz^j,$$

or simply

$$\nabla_i \phi_j = \frac{\partial \phi_j}{\partial z^i} - \sum_{k=1}^n \Gamma^k_{ij} \phi_k.$$

• From now on, we always assume that $D$ is the Hermitian connection associated to the given Hermitian metric (it uniquely exists). We also assume $M$ is Kahler.

• In terms of local coordinate $(z^1, \ldots, z^n)$, we write

$$D^2 \frac{\partial}{\partial z^i} = \sum_{j=1}^n \Omega^i_j \frac{\partial}{\partial z^j},$$

where $\Omega = (\Omega^i_j)$ is called the curvature matrix. We have $\Omega = d\omega - \omega \wedge \omega = \partial(g \cdot g^{-1}) = \partial(\omega)$. Write $\Omega \cdot G = (\Omega_{ik})$, where $G = (g_{ij})$.

• Write $\Omega^i_j = R^i_{jk} d\bar{z}^k \wedge dz^l = R^i_{jkl} dz^l \wedge d\bar{z}^k$ and $\Omega_{ij} := \sum_s g_{si} \Omega^s_j = R_{ijkl} d\bar{z}^k \wedge dz^l$. $R_{ijkl}$ are called the curvature tensors, and $R_{kl} := R_{ijkl} g^{ij}$ are called the Ricci tensors of the Hermitian manifold $M$, where $g^{ij}$ are the entries of the inverse of metric matrix $g$. 

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• From $\Omega = \bar{\partial}(\partial \omega)$,

$$\Omega^i_j = \bar{\partial}(\sum \Gamma^i_{jl} dz^l) = \sum \bar{\partial}_k \Gamma^i_{kl} dz^k \wedge dz^l.$$ 

Hence

$$R^i_{jkl} = \bar{\partial}_k \Gamma^i_{kl}.$$ 

• From $\Omega = \bar{\partial}(\partial g \cdot g^{-1})$, we have

$$R_{jka\bar{b}} = -\sum \frac{\partial^2 g_{jk}}{\partial z^a \partial \bar{z}^b} + \sum g_p^q \frac{\partial g_{jp}}{\partial z^a} \frac{\partial g_{jq}}{\partial \bar{z}^b}$$

and

$$R_{ij} = \partial_i \partial_j (\log \det g).$$

• We also have

$$R_{ij\bar{k}l} = R_{i\bar{k}jl} = R_{i\bar{k}ij} = R_{ij\bar{k}} = R_{\bar{i}j\bar{k}}.$$

2 Hermitian Line and vector bundles

The above concepts can be extended from the tangent bundle $T^{(1,0)}(M)$ to a general vector bundle.

• A holomorphic vector bundle $E$ over $M$ is a topological space together with a continuous mapping $\pi : E \rightarrow M$ such that (i) $E_p = \pi^{-1}(p)$; $\forall p \in M$, is a linear space with rank $r$; (ii) There exists an open covering $\{U_\alpha\}_{\alpha \in I}$ of $M$ and biholomorphic maps $\phi_\alpha$ with

$$\phi_\alpha : \pi^{-1}(U_\alpha) \sim U_\alpha \times \mathbb{C}^r, \quad \forall \alpha \in I$$

and such that

$$\phi_\alpha : E_p \sim \{p\} \times \mathbb{C}^r \sim \mathbb{C}^r, \quad \forall p \in U_\alpha$$

is a $\mathbb{C}$ linear isomorphism between complex vector space. On $U_\alpha \cap U_\beta \neq \emptyset$, let $\phi_{\alpha \beta} := \phi_\alpha \circ \phi^{-1}_\beta$, then $\phi_{\alpha \beta} : U_\alpha \cap U_\beta \rightarrow \text{GL}(r, \mathbb{C})$ is holomorphic. $\phi_{\alpha \beta}$ is called transitive function of $E$; The $\phi_{\alpha \beta}$ satisfies the compatible conditions: $\phi_{\alpha \beta}(p) \phi_{\beta \gamma}(p) = \phi_{\alpha \gamma}(p)$ and $\phi_{\alpha \beta}(p) = \phi_{\beta \alpha}(p)^{-1};$ $p \in U_\alpha \cap U_\beta.$
• The rank 1 vector bundle $L$ is called the line bundle. In this case, on $U_\alpha \cap U_\beta \neq \emptyset$, for $\forall p \in U_\alpha \cap U_\beta$, let $\phi_{\alpha\beta}(p) := \phi_{\alpha} \circ \phi_{\beta}^{-1} \res L_p : C \xrightarrow{\phi_{\beta}^{-1}} L_p \xrightarrow{\phi_{\alpha}} C$, is a $C$-linear isomorphism, it is represented by $g_{\alpha\beta}(p) \in GL(1, C) \cong C^*$, i.e. $\phi_{\alpha\beta}(p, z) = (p, g_{\alpha\beta}(p)z)$. Furthermore $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(1, C) \cong C^*$ is holomorphic. $g_{\alpha\beta}$ is called transitive function of $L$. So $L$ corresponds to $\{U_\alpha, g_{\alpha\beta}\}$ where $\{U_\alpha\}$ is a finite cover of $M$ and $g_{\alpha\beta}$ is nowhere zero holomorphic functions with the compatible conditions.

• Let $e_{\alpha}(p) = \phi^{-1}(p, 1)$. Then $e_{\alpha}$ is a local frame of $L$ on $U_\alpha$. It is called the canonical frame. It is easy to check that $e_{\beta} = g_{\alpha\beta}e_{\alpha}$.

• A (holomorphic) section of $L$ is a holomorphic map $s : M \to L$ such that $\phi \circ s = id$. Write $s = s_{\alpha}e_{\alpha}$ on $U_\alpha$. Then $s_{\alpha} = g_{\alpha\beta}s_{\beta}$.

• Consider $O_{P^n}(-1)$, tautological line bundle on $P^n(C)$ (which some books called it the universal line bundle). It can be described as follows: The set $O_{P^n}(-1) \subset P^n \times C^{n+1}$ that consists of all pairs $(l, z) \in P^n \times C^{n+1}$ with $z \in l$ (i.e. $l = [z]$) forms a natural way a holomorphic line bundle over $P^n$. In fact, let $P^n = \bigcup_{i=0}^{n} U_i$ be the standard open covering. A canonical trivialization of $O_{P^n}(-1)$ over $U_i$ is given by $\psi_i : \pi^{-1}(U_i) \to U_i \times C$, $(l, z) \mapsto (l, z_i)$. Since $\psi_j^{-1}(l, 1) = (l, z_i/z_j)$, we have $\psi_i \circ \psi_j^{-1}(l, 1) = \psi_i(l, z_i/z_j)$ Hence transition maps $\psi_{ij}(l) : C \to C$ are given by $w \mapsto \frac{z_i}{z_j}w$, here we used the fact that $\psi_{ij}(l)$ is linear, where $l = [z_0 : \cdots : z_n]$, i.e. $\psi_{ij}([z]) = \frac{z_i}{z_j}$. Note that the fiber $O_{P^n}(-1)$ over $P^n$ is naturally isomorphic to $l$.

• The line bundle of hyperplane of $P^n$ : The dual of $O_{P^n}(-1)$, denoted by $O_{P^n}(1)$ is called the hyperplane line bundle. Its transition functions are $g_{\alpha\beta} = \frac{z^{\alpha}}{z^{\beta}}$. On $U_\alpha$, consider $s_{\alpha} = a_1z_1 + \cdots + a_{n-1}z^{n-1}_{\alpha} + a_{\alpha} + a_{\alpha+1}z^{\alpha+1}_{\alpha} + \cdots + a_nz^n_{\alpha}$. Then $s_{\alpha} = \frac{z^\alpha}{z^\beta}s_{\beta}$. So $s_{\alpha}$ defined a holomorphic section $s = a_0z_0 + \cdots + a_nz^n$. It zero is the hyperplane $H = \{[z^0, \cdots, z^n] \in P^n | \sum_{\alpha=0}^{n} a_{\alpha}z^\alpha = 0\}$ in $P^n$. This is where the name of hyperplane line bundle of $P^n$ comes from. We sometimes also denoted it by $[H]$.

• $E$ is called a Hermitian vector bundle if there is an Hermitian inner product on each fiber $E_p$ for $p \in M$. 


• Similar to above, with the given Hermitian metric, there is a canonical connection (called Hermitian connection) \( D : \Gamma(M, E) \to \Gamma(M, \Lambda^1(M) \otimes E) \) which is compatible with the complex structure and with the Hermitian metric on \( E \).

• For simplicity, we only focus on the line bundle \( E = L \), i.e. \( r = 1 \). Let \( \{U_\alpha\}_{\alpha \in I} \) be trivialization neighborhoods of \( L \). Then the Hermitian metric \( \{h_\alpha := h(e_\alpha, e_\alpha)\}_{\alpha \in I} \) is a set of positive functions with \( h_\alpha = |g_\beta\alpha|^2 h_\beta \) on \( U_\alpha \cap U_\beta \), where \( g_\beta\alpha \) are transition functions. Its connection form is

\[
\theta = \partial h_\alpha \cdot h_\alpha^{-1} = \partial \log h_\alpha,
\]

and the curvature form is

\[
\Theta = \bar{\partial} \partial \log h_\alpha = \bar{\partial} \partial \log h_\beta, \text{ on } U_\alpha \cap U_\beta.
\]

So \( \Theta \) is a global (1,1)-form on \( M \). Define the first Chern form of the Hermitian line bundle \( (L, h) \) as \( c_1(L, h) = \frac{i}{2\pi} \Theta = \frac{i}{2\pi} \bar{\partial} \partial \log h_\alpha \).

• \((L, h)\) is said to be positive (or ample) if \( c_1(L, h) \) is positive.

• On the hyperplane line bundle of hyperplane line bundle of \( \mathbb{P}^n \). We endow with a Hermitian metric \( h \) on line bundle \( [H] \), \( h = (h_\alpha)_{0 \leq \alpha \leq n} \), where \( h_\alpha \) is the local expression of \( h \) on \( U_\alpha \).

\[
h_\alpha = \frac{|z^\alpha|^2}{|z|^2} = \frac{1}{\sum_{\beta \neq \alpha} |\bar{z}^\beta|^2 + 1}.
\]

\[
c_1([H]) = -\frac{\sqrt{-1}}{2\pi} \bar{\partial} \partial \log h_\alpha = \frac{\sqrt{-1}}{2\pi} \bar{\partial} \partial \log ||z||^2 > 0.
\]

so \([H]\) is positive line bundle. It is easy to see that \([H]\) is, in fact, independent of the choice of \( H \), so we denote it by \( \mathcal{O}_{\mathbb{P}^n}(1) \).

• A divisor \( D \) on \( M \) is a formal linear combination

\[
D = \sum n_i [Y_i]
\]

where \( Y_i \subset M \) irreducible hypersurfaces and \( n_i \) are integers. A divisor \( D \) is called effective if \( n_i \geq 0 \) for all \( i \).
• Any divisor $D$ induces $\mathcal{O}(D)$, the line bundle associated to $D$, in a canonical way: If $D$ is a hypersurface locally defined by $f_\alpha = 0$ on $U_\alpha$, then $\phi_{\alpha\beta} = f_\alpha/f_\beta$ are the transition functions for $\mathcal{O}(D)$. If $D = H$ is a hyperplane, then $\mathcal{O}(H) = \mathcal{O}_{\mathbb{P}^n}(1)$.

• **Canonical line bundle on $M$:** Let $\{U_\alpha\}_{\alpha \in I}$ be a holomorphic coordinate covering of $M$, $(z^1_\alpha, \ldots, z^n_\alpha)$ be a local coordinate system of $U_\alpha$. The canonical line bundle $K_M$ is the line bundle with the transition functions $\phi_{\alpha\beta} = \det \frac{\partial (z^j_\beta)}{\partial (z^i_\alpha)}$. Sections of $K_M$ are $(n, 0)$-forms $\omega = a^\alpha dz^1_\alpha \wedge \cdots \wedge z^n_\alpha$.

With Hermitian metric $ds^2 = g^{(\alpha)}_{ij} dz^i_\alpha \otimes dz^j_\alpha$ on $M$, $\det g^{(\alpha)} = \det (g^{(\alpha)})$ is an Hermitian metric of $\det(T^{1,0}(M))$, thus $\det g^{(\alpha)^{-1}}$ is the Hermitian metric of $K_M$. The connection form of $K_M$ is thus $\theta^{(\alpha)} = \partial \log \det g^{(\alpha)}$, and the curvature form is $\Omega^{(\alpha)} = -\partial \bar{\partial} \log \det g^{(\alpha)} = R_{ji} dz^i \wedge d\bar{z}^j$.

### 3 Sheaves and Cohomology

• **Origins. The Mittag-Leffler Problem:** Let $S$ be a Riemann surface, not necessarily compact, $p \in S$ with local coordinate $z$ centered at $p$. A principal part at $p$ is the polar part $\sum_{k=1}^\infty a_k z^{-k}$ of Laurent series. If $\mathcal{O}_p$ is the local ring of holomorphic functions around $p$, $\mathcal{M}_p$ the field of meromorphic functions around $p$, a principal is just an element of the quotient group $\mathcal{O}_p/\mathcal{M}_p$. The Mittag-Leffler question is, given a discrete set $\{p_n\}$ of points in $S$ and a principal part at $p_n$ for each $n$, does there exist a meromorphic function $f$ on $S$, holomorphic outside $\{p_n\}$, whose principal part at each $p_n$ is the one specified? The question is clearly trivial locally, and so the problem is one of passage from local to global data. Here are two approaches, both lead to cohomology theories.

Cech: Take a covering $\{U_\alpha\}$ of $S$ by open sets such that each $U_\alpha$ contains at most one point $p_n$, and let $f_\alpha$ be a meromorphic function on $U_\alpha$ solving the problem in $U_\alpha$. Set

$$f_{\alpha\beta} = f_\alpha - f_\beta \in \mathcal{O}(U_\alpha \cap U_\beta).$$
In $U_\alpha \cap U_\beta \cap U_\gamma$, we have

$$f_{\alpha \beta} + f_{\beta \gamma} + f_{\gamma \alpha} = 0.$$  

Solving the problem globally is equivalent to finding $\{g_\alpha \in \mathcal{O}(U_\alpha)\}$ such that $f_{\alpha \beta} = g_\beta - g_\alpha$ in $U_\alpha \cap U_\beta$: given that $g_\alpha$, $f = f_\alpha + g_\alpha$ is a globally defined function satisfying the conditions, and conversely. In the Čech theory,

$$Z^1([U_\alpha], \mathcal{O}) = \{\{f_{\alpha \beta}\} : f_{\alpha \beta} + f_{\beta \gamma} + f_{\gamma \alpha} = 0\}$$

and the first Čech cohomology group

$$H^1([U_\alpha], \mathcal{O}) = Z^1([U_\alpha], \mathcal{O})/\delta C^0([U_\alpha], \mathcal{O})$$

is the obstruction to solving the problem.

**Dolbault:** As before, take $f_\alpha$ be a meromorphic function on $U_\alpha$ solving the problem in $U_\alpha$, and let $\rho_\alpha$ be a bump function, 1 in a neighborhood of $p_n \in U_\alpha$ and having compact support in $U_\alpha$. Then

$$\phi = \sum_\alpha \overline{\partial}(\rho_\alpha f_\alpha)$$

is a $\overline{\partial}$-closed $c^\infty$-(0,1)-form on $S$ ($\phi \equiv 0$ in a neighborhood of $p_n$). If $\phi = \partial \eta$ for $\eta \in C^\infty(S)$, then the function

$$f = \sum_\alpha \rho_\alpha f_\alpha - \eta$$

satisfies the conditions of the problem: thus the obstruction to solving the problem is in $H^{0,1}_{\text{Dol}}(S)$.

As the theorem of Dolbault, these two co-homology groups are the same (isomorphism).

- **A Sheaf** $\mathcal{F}$ over a complex manifold $X$ consists of, for each open set $U \subset X$, an abelian group (or vector spaces, rings, or any desired object) $\mathcal{F}(U)$ (also denoted $\Gamma(\mathcal{F}, U)$ and called the set of sections over $U$),
and a collection of restriction maps such that for each $U \subset V \subset X$, 
\( \rho_{V,U} : F(V) \rightarrow F(U) \), and satisfy:

1. **Identity**: $\rho_{U,U} = id|_{F(U)}$,

2. **Compatibility**: If $U \subset V \subset W \subset X$, then $\rho_{V,U} \circ \rho_{W,V} = \rho_{W,U}$;

3. **Sheaf axiom (gluing)**: Let $U = U_{p} \cup_{\alpha} U_{\alpha}$ and $\sigma_{\alpha}|_{U_{\alpha} \cap U_{\beta}} = \sigma_{\beta}|_{U_{\alpha} \cap U_{\beta}}$ for all $\alpha, \beta$, then there exists a (unique) $\sigma \in F(U)$ such that $\sigma_{\alpha} = \sigma|_{\alpha}$ for all $\alpha$.

If only (1) and (2) are satisfied, then $F$ is call a **presheaf**.

- **Examples**: Let’s us consider a few familiar examples, each with the natural sections and restriction maps: $O_{X}$, the sheaf of holomorphic functions on $X$; $\Omega^{p}_{X}$, the sheaf of holomorphic $p$-forms on $X$; $A^{n}_{X}$, the sheaf of $n$-forms on $X$; $A^{a,b}_{X}$, the sheaf of $(a,b)$-forms on $X$; The skyscraper sheaf $C_{p}$ given by $C_{p}(U) = C$ if $p \in U$, and $C_{p}(U) = 0$ if $p \not\in U$ along with the natural restriction maps.

- **Čech cohomology**: Let $\mathcal{F}$ be an abelian group sheaf over a complex manifold $X$. Let $\mathcal{U} = \{U_{i}\}_{i \in I}$ be an open covering of topological space $X$. For $\forall$ nonnegative integer $p$, consider

  \[ f : \{ |\sigma| \cap U_{i_{0}} \cap \ldots \cap U_{i_{p}} \neq \emptyset \} \rightarrow \Gamma(|\sigma|, \mathcal{F}) : U_{i_{0}} \cap \ldots \cap U_{i_{p}} \rightarrow f_{i_{0}\ldots i_{p}}. \]

If $U_{i_{0}} \cap \ldots \cap U_{i_{p}} \emptyset$, we take $f_{i_{0}\ldots i_{p}} = 0$. Such $f = \{f_{i_{0}\ldots i_{p}}\}$ is called a $p$-cochain of $\mathcal{U}$ with coefficients in the sheaf $\mathcal{F}$. We use $C^{p}(\mathcal{U}, \mathcal{F})$ to denote the set of all $p$-cochains of $\mathcal{U}$ with coefficients in the sheaf $\mathcal{F}$. For $\forall \{f_{i_{0}\ldots i_{p}}\}, \{g_{i_{0}\ldots i_{p}}\} \in C^{p}(\mathcal{U}, \mathcal{F})$, defining the addition operation

\[ \{f_{i_{0}\ldots i_{p}}\} + \{g_{i_{0}\ldots i_{p}}\} = \{f_{i_{0}\ldots i_{p}} + g_{i_{0}\ldots i_{p}}\} \]

then $C^{p}(\mathcal{U}, \mathcal{F})$ becomes an abelian group, we called $C^{p}(\mathcal{U}, \mathcal{F})$ $p$-dimensional **cochains group** of $\mathcal{U}$ with **coefficients in sheaf** $\mathcal{F}$.

Now we define the operator

\[ \delta_{p} : C^{p}(\mathcal{U}, \mathcal{F}) \rightarrow C^{p+1}(\mathcal{U}, \mathcal{F}) : f \rightarrow \delta_{p}f \]

where

\[ (\delta_{p}f)_{i_{0}\ldots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^{k} f_{i_{0}\ldots \hat{i_{k}} \ldots i_{p+1}}. \]
In the right hand side of (1), each \( f_{i_0 \ldots i_p+1} \) restricts to \( U_{i_0} \cap \cdots \cap U_{i_p+1} \) and proceeds the addition operation in \( \Gamma(U_{i_0} \cap \cdots \cap U_{i_p+1}, \mathcal{F}) \). It is easy to verify \( \delta_p \) is a homeomorphism of group, and \( \delta_{p+1} \circ \delta_p = 0; \ p \geq 1 \). \( Z^p(U, \mathcal{F}) := \ker \delta_p \subset C^p(U, \mathcal{F}), \ p \geq 0, \) is called the \( p \)-dimensional cocycles group of \( U \) with coefficients in sheaf \( \mathcal{F} \), and \( B^p(U, \mathcal{F}) = \text{Im } \delta_{p-1}, \ p \geq 1, \) is called the \( p \)-dimensional coboundaries group of \( U \) with coefficients in sheaf \( \mathcal{F} \), and \( B^0(U, \mathcal{F}) \equiv 0 \). From \( \delta_{p+1} \circ \delta_p \equiv 0, \ B^p(U, \mathcal{F}) \subset Z^p(U, \mathcal{F}). \)

\[ H^p(U, \mathcal{F}) = \left\{ \begin{array}{ll} Z^p(U, \mathcal{F})/B^p(U, \mathcal{F}), & p \geq 1 \\ Z^0(U, \mathcal{F}), & p = 0 \end{array} \right. \]

\( H^p(U, \mathcal{F}) \) is called the \( p \)-dimensional cohomology group of \( U \) with coefficients in the sheaf \( \mathcal{F} \). Define \( H^p(X, \mathcal{F}) = \lim_{\to} H^p(U, \mathcal{F}). \)

- Let \( C_p \) be the skyscraper sheaf. Then (i) \( H^0(M, C_p) = C \), (ii) \( H^1(M, p) = 0 \). The assertion of (i) is trivial. As for (ii), consider a cohomology class \( \xi \in H^1(M, C_p) \), which is represented by a cocycle in \( Z(U, C_p) \). The covering \( \mathcal{U} \) has a refinement \( \mathcal{B} = \{ V_\alpha \} \) such that the point \( p \) is contained in only one \( V_\alpha \). But then \( Z(U, C_p) = 0 \) and hence \( \xi = 0 \). This finishes the proof.

- **The Dolbeault Theorem.** Recall the define of the Dolbeault cohomology

\[ H^{a,b}_{\text{Dol}}(M) = \ker \bar{\partial}_{a,b}/\text{im } \bar{\partial}_{a,b-1}. \]

Let \( U \) be an open subset of \( M \), denote by

\[ \Omega^p_M(U) := \Gamma(U, \Omega^p_M) := \{ \omega \in \mathcal{A}^{p,0}(U), \bar{\partial} \omega = 0 \}, \]

the set of holomorphic \( p \)-forms on \( U \). \( \Omega^p_M \) is the sheaf of holomorphic \( p \)-forms. Also let

\[ Z^{p,q}_{\overline{\partial}}(U) := \{ \omega \in \mathcal{A}^{p,q}(U), \bar{\partial} \omega = 0 \}, \]

the set of holomorphic \( (p,q) \)-forms on \( U \). \( Z^{p,q}_{\overline{\partial}} \) is the sheaf of holomorphic \( (p,q) \)-forms. The following is the statement of the Dolbeault Theorem: Let \( X \) be a compact complex manifold. Then

\[ H^{p,q}_{\text{Dol}}(M) = H^q(M, \Omega^p_M), \]
where $\Omega^p_M$ is the sheaf of holomorphic $p$-forms.

• The proof of Dolbeault Theorem uses the following fact that, by the $\bar{\partial}$-Poincare lemma, the sequence, under $\bar{\partial}$,

$$0 \to \Omega^p \to A^{p,0} \to A^{p,1} \to A^{p,2} \to \cdots$$

is exact on any complex manifold, as well as the theorem about that the "short" exact sequence about sheaves induces the long exact sequence about the co-homologies.

4 Hodge Theory

Suppose $M$ is compact complex manifold of complex dimension $n$ and $E$ is a holomorphic bundles of rank $r$. From the Dolbeault isomorphism we know that the cohomology group $H^q(M, E)$ is isomorphic to the set of all $\bar{\partial}$-closed smooth $E$-valued $(0, q)$-forms on $M$ modulo the set of all $\bar{\partial}$-exact smooth $E$-valued $(0, q)$-forms on $M$ (if you are not familiar with the sheaf theory, you can regard it as the definition of $H^q(M, E)$). Though this gives us a more practical way of computing $H^q(M, E)$ by using differential forms, it would be more convenient for computational purpose if a cohomology class is represented by a unique differential form rather than an equivalence class of differential forms. The Hodge theorem states that such is case if $M$ is compact Kahler, i.e. every equivalence class of differential forms is uniquely represented by the harmonic differential form (which is unique).

Here the concept of harmonic is defined as follows: The Kahler metric $g$ on $M$ and Hermitian metric $h$ on $E$ together induce a (canonical) inner product on the space of smooth $E$-valued $(p, q)$-forms as follows:

$$\langle \phi, \psi \rangle = \phi_{1\bar{j}1} \bar{\psi}_{K\bar{L}} h^{i\bar{i}k_i} \cdots g^{1\bar{1}r_1} g^{1\bar{1}j_1} \cdots g^{1\bar{1}j_q}.$$ 

It is easy to check that it is independent of all the frames that were chosen. When we want to indicate the dependence on the metrics $g$ and $h$, we shall write $\langle \phi, \eta \rangle_{g,h}$. The (global) inner product is defined as

$$(\phi, \psi) = \frac{1}{plql} \int_M \langle \phi, \psi \rangle_{g,h} dV,$$
where \(dV\) is the volume form. Denote by \(\bar{\partial}^*\) the adjoint of \(\bar{\partial}\) with respect to the global inner product defined above. \(\Box := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}\) is called the complex Laplacian. An \(E\)-valued form \(\phi\) is called harmonic if \(\Box \phi = 0\).

## 5 Kodaira’s vanishing theorem

- The vanishing theorem of Kodaira says that the cohomology groups \(H^q(M, L \otimes K_M)\) vanishes for \(q \geq 1\) when \(L\) is a holomorphic line bundle over a compact complex Kahler manifold \(M\) and \(L\) admits a Hermitian metric whose curvature form is positive definite. Here \(K_M\) is the canonical line bundle of \(M\).

- The proof of Kodaira’s vanishing theorem relies on the following Bochner-Kodaira’s formula: for any smooth \(E\)-valued \((0,q)\)-form \(\phi\),

\[
(\Box \phi, \phi) = \left\| \bar{\nabla}\phi \right\|^2_M - (Ric \phi, \phi)_M + (\Omega \phi, \phi)_M,
\]

where

\[
\left\| \bar{\nabla}\phi \right\|^2_M = \int_M g^{i\bar{j}} \bar{\nabla}_j \phi_{i\bar{\alpha}} \bar{\nabla}_{\bar{i}} \phi_{j\bar{\alpha}}^{1\cdots q},
\]

\[
(Ric \phi, \phi)_M = \sum_{k=1}^{q} \int_M R^k_{j_k \bar{j}_1 \cdots \bar{j}_q} \phi^{i_{\bar{k}}} \phi_{i_{\bar{k}}^{1\cdots q}},
\]

\[
(\Omega \phi, \phi)_M = \sum_{k=1}^{q} \int_M \Omega^k_{j_k \bar{j}_1 \cdots \bar{j}_q} \phi^{i_{\bar{k}}} \phi_{i_{\bar{k}}^{1\cdots q}},
\]

where \(\Omega\) is the curvature form of \(E\).

- Note that the curvature form on \(K_M\) is given by \(R_{j\bar{i}} dz^j \wedge d\bar{z}^\bar{i}\). Take \(E = L \otimes K_M\). If \(L\) is positive, i.e. the Hermitian matrix \(\Omega_{i\bar{j}} - R_{i\bar{j}}\) is positive definite at each point, then we will have \(\Box \phi, \phi \geq 0\). This implies that \(\phi = 0\) if \(\phi\) is harmonic. Thus \(H^q(M, L \otimes K_M) = 0\) for \(q \geq 1\) by the Hodge theorem.

- Assuming \(E\) is trivial. Then the calculation of \(\Box\) depends on the following local formula for \(\bar{\partial}\) and \(\bar{\partial}^*\) for \(\phi = \sum \phi_{I_p \bar{J}_q} \otimes dz^{I_p} \wedge d\bar{z}^{\bar{J}_q}\),

\[
(\bar{\partial} \phi)_{I_p \bar{J}_0 \bar{J}_1 \cdots \bar{J}_q} = (-1)^p \sum_{k=0}^{q} (-1)^k \bar{\nabla}_{j_k} \phi_{I_p \bar{J}_0 \bar{J}_1 \cdots \bar{J}_k \cdots \bar{J}_q}.
\]

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and
\[(\bar{\partial}^* \phi)_{I_p j_1 \cdots j_{q-1}} = (-1)^{p+1} g^{j_i j_1 \cdots j_{q-1}},\]
as well as the computation of \([\nabla_i, \nabla_j]\) for \((1,0)\) \(a_k dz^k\) and \((0,1)\)-forms \(b_k d\bar{z}^k\) as follows:
\[
[\nabla_i, \nabla_j] a_k = R^t_{kj} a_t,
\]
\[
[\nabla_i, \nabla_j] b_k = -R^t_{kj} b_t.
\]

In general (when \(E\) is not trivial with metric \(h_\alpha\)), then the connection induced by \(h_\alpha\) is \(D_E : \Gamma(M, \mathcal{E}^{p,q}(E)) \to \Gamma(M, \mathcal{E}^{p,q+1}(E))\), which is given by
\[
D_E \omega_\alpha = \partial \omega_\alpha + (\partial \log h_\alpha) \wedge \omega_\alpha.
\]

So if we write \(D_E = dz^i \nabla_i^E\) then \(\nabla_i^E \omega\) is still an \(E\)-valued \((p,q)\)-form. We call \(\nabla_i^E; 1 \leq i \leq n\), the covariant derivatives with respect to Hermitian line bundle \(E\). Similar to above, for \(\forall \eta \in \Gamma(M, \mathcal{E}^{p,q+1}(E))\),
\[
(\bar{\partial}_E \eta_{i_1 \cdots i_p j_1 \cdots j_q}) = (-1)^{p+1} g^{j_i j_1 \cdots j_{q-1}} \nabla_i^E \eta_{i_1 \cdots i_p j_1 \cdots j_{q-1}}.
\]

### 6 Singular metric, currents, Nadel’s multiplier ideal sheave and Nadel’s vanishing theorem

- **Kodaria’s vanishing theorem** has been extended by Nadel to line bundles with singular metric (i.e. \(h = \{h_\alpha\}\), where \(h_\alpha\) may be singular). We write \(h_\alpha = e^{-\kappa_\alpha}\), here we usually write \(\kappa := \kappa_\alpha\) if no risk of confusion, then the curvature \(\Theta_h := \partial \bar{\partial} \kappa\) is not a smooth differential form anymore if the metrics singular(it is in fact called current). We say that \(e^\kappa\) has non-negative (reps. positive) curvature current if \(\Theta_h\) is a non-negative (reps. \((1,1)\)- current, or equivalently, the local representatives \(\kappa\) are plurisubharmonic.

- **Currents:** Recall that if \(f, g \in C^0[0,1]\), then \(f \equiv g\) if and only if \(\int_0^1 f(x) \phi(x) = \int_0^1 g(x) \phi(x)\) for every \(\phi \in C^\infty_0[0,1]\). Also for closed intervals \(A, B \subset \mathbb{R}\), \(A = B\) if and only if \(\int_A \phi(x) = \int_B \phi(x)\). Here "functions" and "subsets" can be regarded as linear functional forms on
These concepts are unified by a general concept of currents: Let $M$ be a real differentiable manifold with $\dim M = m$. A current of degree $q = m - p$ (or dimension $p$) is a real linear map $T : \mathcal{D}^p(M) \to \mathbb{R}$, where $\mathcal{D}^p(M)$ is the set of smooth $p$-forms on $M$ with compact support. Typical examples are smooth or $L^1_{\text{loc}}$-forms $\beta$ with $T = [\beta]$ defined by $T(\phi) = \int_M \beta \wedge \phi$ for $\phi \in \mathcal{D}^p(M)$ with $q = m - p$, as well as $p$-dimensional oriented submanifold $S \subset M$ with $T = [S]$ defined as $T(\phi) = \int_S \phi$.

- **Poincare-Lelong formula:** Let $f \in \mathcal{O}(M)$ be a holomorphic function. Then
  \[
  \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |f|^2 = [f = 0]
  \]
  holds as currents.

- **Example of singular metric:** Let $L \to M$ be a holomorphic line bundle. Let $m$ be a positive integer and $s_1, \ldots, s_N$ be sections of $mL$. Write $s = s_\alpha e^\alpha$, and define
  \[
  \kappa_\alpha = \frac{1}{m} \log(|s^1_\alpha|^2 + \cdots + |s^N_\alpha|^2).
  \]
  This singular metric blows up exactly on the common zeros of the sections $s^1, \ldots, s_N$.

- **Let** $U \subset M$ be an open subset, and let $\phi$ be a locally integrable function on $U$. We define
  \[
  \mathcal{I}(U) := \{ f \in \mathcal{O}_M(U) : |f|^2 e^{-\phi} \in L^1_{\text{loc}}(U) \}.
  \]
  The corresponding sheaf of germs $\mathcal{I}_\phi$ is called the multiplier ideal sheaf associated to $\phi$.

- **Nadel proved** that if $\phi$ is a plurisubharmonic, then the multiplier ideal sheaf $\mathcal{I}_\phi$ is a coherent sheaf of ideals.

- **Nadel’s Vanishing Theorem:** Let $X$ be a compact Kähler manifold with Kähler form $\omega$, and let $F$ be a line bundle with singular Hermitian metric $h = e^{-\phi}$ such that $\sqrt{-1} \partial \bar{\partial} \phi \geq \epsilon \omega$ for some continuous function
\( \epsilon > 0 \) (in the sense of distribution). Then, for \( q \geq 1 \), \( H^q(X, \mathcal{O}_X(K_X + \mathcal{F}) \otimes I_\phi) = 0 \) where \( K_X \) is the canonical line bundle of \( X \).

The point of the proof is that any plurisubharmonic function is the limit of a decreasing sequence of smooth plurisubharmonic functions, so eventually it can be reduced to the smooth case.

- Lelong numbers of plurisubharmonic functions: The zero of the ideal sheaf \( I_\phi \) is then the set of points where \( e^{-\phi} \) is not locally integrable. Such points only occur where \( \phi \) has poles, but the poles need to have a sufficiently high order. If \( \phi = \frac{1}{m} \log(|s_1|^2 + \cdots + |s_n|^2) \) as in the example earlier, then one has a notion of (log)-pole order. In general, the pole orders are defined using the so-called Lelong numbers: Let \( X \) be a complex manifold and \( \phi \) a plurisubharmonic function in a neighborhood \( U \) of \( x \in X \). Fix a coordinate chart \( U \) near \( x \), and let \( z \) be a local coordinates vanishing on \( x \). The Lelong number of \( \phi \) is defined to be the number
  
  \[
  v(\phi, x) := \liminf_{z \to x} \frac{\phi(z)}{\log |x - z|^2}.
  \]

  We also set
  
  \[
  E_c(\phi) := \{ x \in X ; v(\phi, x) \geq c \}.
  \]

- A famous paper of Siu showed that \( E_c(\phi) \) is a complex analytic set.

- The Lelong number information \( v(\phi, x) \) gives the information about the vanishing order of \( f \) at \( x \) for \( f \in \mathcal{I}_{\phi,x} \) which is stated as the lemma of Skoda: Let \( \phi \) a plurisubharmonic function on an open set \( U \) of \( X \) containing \( x \). Then (1). If \( v(\phi, x) < 1 \), then \( e^{-\phi} \) is integrable in a neighborhood of \( x \). In particular, \( \mathcal{I}_{\phi,x} = \mathcal{O}_{U,x} \); (2). If \( v(\phi, x) \geq n + s \) for some positive integer, then the estimate \( e^{-\phi} \geq C|x - z|^{-(n+s)} \) holds in a neighborhood of \( x \). In particular, one obtains that \( \mathcal{I}_{\phi,x} \subset \mathcal{m}_{U,x}^{n+1} \), where \( \mathcal{m}_{U,x} \) is the maximal ideal of \( \mathcal{O}_{G,x} \); 3. The zero variety \( V(\mathcal{I}_\phi) \) of \( \mathcal{I}_\phi \) satisfies \( E_{2n}(\phi) \subset V(\mathcal{I}_\phi) \subset E_2(\phi) \).

- Nadel’s vanishing theorem plus Skoda’s lemma gives a new proof (without using blow-ups) of Kodaira’s embedding theorem: Let \( X \) be a compact Kähler manifold. Assume there exists a positive line bundle \( L \) over \( X \), then \( X \) can be embedded in projective space \( \mathbb{P}^N \)
To prove the embedding theorem, it gets down to construct holomorphic sections. Consider the long exact sequence of cohomology associated to the short exact sequence
\[ 0 \to \mathcal{I}_\phi \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}_\phi \to 0 \]
twisted by \( \mathcal{O}(K_X \otimes L) \), and apply Nadel’s vanishing theorem of the first \( H^1 \) group, we’ll have: Let \( X \) be a weakly pseudo-convex Kahler manifold with Kahler form \( \omega \), and let \( F \) be a line bundle with singular Hermitian metric \( h = e^{-\phi} \) such that \( \sqrt{-1} \partial \bar{\partial} \phi \geq \epsilon \omega \) for some continuous function \( \epsilon > 0 \). Let \( x_1, \ldots, x_N \) be isolated points in the zero variety \( V(\mathcal{I}_\phi) \). Then there is a surjective map
\[ H^0(X, K_X \otimes L) \to \bigoplus_{1 \leq j \leq N} \mathcal{O}(K_X \otimes L)_{x_j} \otimes (\mathcal{O}_X/\mathcal{I}_\phi)_{x_j}. \]

Exercise: Assume that \( X \) is compact and \( L \) is a positive line bundle. Let \( \{x_1, \ldots, x_N\} \) be a finite set. Show that there are constants \( a, b \geq 0 \) depending only on \( L \) and \( N \) such that \( H^0(X, L^\otimes M) \) generates jets of any order \( s \) at all points \( x_j \) for \( m \geq as + b \).

**Hint.** Apply the above Corollary to \( L' = K_X^{-1} \otimes L^\otimes m \), with a singular metric on \( L \) of the form \( h = h_0 e^{-\psi} \), where \( h_0 \) is smooth of positive curvature, \( \epsilon > 0 \) small and
\[ \psi(z) = \sum \chi_j(z)(n + s - 1) \log \sum |w^{(j)}(z)|^2 \]
with respect to coordinate systems \( (w^{(j)}_k(z))_{1 \leq k \leq n} \) centered at \( x_j \). The cut-off functions \( \chi_j \) can be taken of a fixed radius (bounded away from 0) with respect to a finite collection of coordinate patches covering \( X \). It is easy to see such \( h \) serves our purposes.

Taking \( s = 2 \) and \( m \) with \( m \geq 2a + b \) as in the Exercise, then the sections of \( H^0(X, L^\otimes m) \) generates any pair of \( L_x \oplus L_y \) for distinct points \( x \neq y \) in \( X \), as well as 1-jets of \( L \) at any point \( x \in X \). The existence of the section of \( H^0(X, L^\otimes m) \) which generates any pair of \( L_x \oplus L_y \) for distinct points \( x \neq y \) in \( X \) implies that \( F \) is injective. Now, we use the
fact that there is a section $s$ of $H^0(X, L^{\otimes m})$ which generates 1-jets of $L$ at any point $x \in X$, i.e. the section $s$ vanishes to the second order. Choosing sections $s^1, \ldots, s^n$ such that the function $s^1/s, \ldots, s^n/s$ have independent differential at $x$, then the holomorphic map

$$\left( \frac{s^1}{s}, \ldots, \frac{s^n}{s} \right)$$

defined in a neighborhood of $x$ is an immersion near $x$. This complete the proof of Kodaira’s imbedding theorem.