What is Number Theory?

Dr. Min Ru, University of Houston

Number theory is the study of the set of positive numbers:

$$1, 2, 3, 4, 5, 6, \ldots.$$

We will especially want to study the *relationship* between different sorts of numbers. Since ancient times, people have separated the whole numbers into variety of different types. Here are some familiar and not-so-familiar examples:

odd	1, 3, 5, 7, 9, 11, . . . ,
even	2, 4, 6, 8, 10, . . . ,
square	1, 4, 9, 16, 25, 36, . . .
cube	1, 8, 27, 64, 125, . . .
prime	2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, . . .
composite	4, 6, 8, 9, 10, 12, 14, 15, 16, . . .
1 (modulo 4)	1, 5, 9, 13, 17, 21, 25, . . .
triangular	1, 3, 6, 10, 15, 21, . . .
perfect	6, 28, 496, . . .

Many of these types of numbers are undoubtedly already known to you. Others may be not be familiar. A number is called triangular if that number of pebbles can be arranged in a triangle, with one pebble at the top, two pebbles in the next row, and so on. A number is perfect if the sum of all of its divisors, other than itself, adds back up to the original number.

Some Typical Number Theoretic Questions

The main goal of number theory is to discover interesting and unexpected relationships between different sorts of numbers and to prove that these relationship are true. In this section we will describe a few typical number theoretic problems, some of which we will eventually solve, some of which have known solutions too difficult for us to include, and some which remain unsolved to this day.

(a) *Sums of Squares I*

Can the sum of two squares be a square? The answer is clearly "YES"; for example $3^2 + 4^2 = 5^2$ and $5^2 + 12^2 = 13^2$. These examples of *Pythagorean triples.*

(b) *Sums of Higher Powers*

Can the sum of two cubes be a cube? Can the sum of two fourth powers be a fourth power? In general, can the sum of two n^{th} powers be an n^{th} powers? The answer is "NO". The famous problem, called *Fermat's Last Theorem*, was first posted by Pierre de Fermat in the 17^{th} century, but was not completely solved until 1994 by Andrew Wiles. Wiles' proof used sophisticated mathematical techniques which we will not be able to describe in detail.

(c) *Infinitude of Primes*

A *prime number* is a number p whose only factors are 1 and p. Are there infinitely many prime numbers? Are there infinitely many primes which are 1 modulo 4 numbers? Are there infinitely many primes which are 3 modulo 4 numbers? The answer is "YES".

(d) *Sums of Squares II*

Which numbers are the sums of two squares? It often turns out that questions of this sort are easier to answer first for primes, so we ask which (odd) prime numbers are a sums of two squares? For example,

3 =NO, $5 = 1^2+2^2$, 7 =NO, 11 =NO, $13 = 2^2+3^2$, $17 = 1^2+4^2$, 19 =NO, 23 =NO, $29 = 2^2 + 5^2$, 31 =NO, $37 = 1^2 + 6^2$, $\ldots$,

Do you see a pattern? Possibly not, since there is only a short list, but a longer list leads to the conjecture that p is a sum of two squares if it is congruent to 1 (modulo 4). That is, p is a sum of two squares if it leaves a remainder of 1 when divided by 4, and it is not a not a sum of two squares if it leave a remainder of 3. This conjecture is true.

(e) *Twin Primes*

In the list of primes it is sometimes true that consecutive off numbers are both prime. For example, among the list of primes less than 100: 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, there are 7 pairs of *twin primes*: 3, 5,7; 11, 13; 17, 19; 29, 13; 41, 43; 59, 61; 71, 73. Are there infinitely many twin primes? That is, are there infinitely many primes p such that $p+2$ is also a prime? At present, on one knows the answer to this question.

(f) *Primes of the Form* $N^2 + 1$

If we list the numbers of the form $N^2 + 1$ taking $N = 1, 2, 3, \ldots$, we find that some of them are prime. Of course, if N is odd, then $N^2 + 1$ is even, so it won't be prime unless $N = 1$. We ask whether there are infinitely many primes of the form N^2+1? Again, on one presently knows the answer to this question.

We have now seen some of the type of questions that are studied in the Theory of Numbers. How are these questions studied? The answer is that Number Theory is partly experimental and partly theoretical. The experimental part normally comes first; it leads to questions and suggestions ways to answer them. The theoretical part follows; in this part, one tries to devise an argument which gives a conclusive answer to the questions, that is a logical sequence of assertions, starting from known facts and ending at the desired statement(we call it a *proof*). This course will follow this path. We will do the experimental part by using the "Web Lab", and then do the theoretical part by providing theorems with proof.