Nevanlinna Theory: Nevanlinna and Cartan's Results and their Generalization

Min Ru

University of Houston

minru@math.uh.edu

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$$\begin{aligned} \log|c_f| &= \int_0^{2\pi} \log|f(Re^{i\theta})| \frac{d\theta}{2\pi} - \sum_{\mu=1}^{p} \log\left|\frac{R}{a_\mu}\right| + \sum_{\nu=1}^{q} \log\left|\frac{R}{b_\nu}\right| \\ &\quad -(\mathrm{ord}_0 f) \log R, \end{aligned}$$

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  where $f(z) = c_f z^{\mathrm{ord}_0 f} + \cdots$, $\mathrm{ord}_0 f \in \mathbf{Z}$, and c_f is the leading nonzero coefficient.

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- The Nevanlinna's proximity function $m_f(r,\infty)$ is defined by

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- FMT(From Jensen's formula):
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- **Theorem**:

$$\begin{aligned} m_{f'/f}(r) &\leq 3\log^+ R + 4\log^+ \frac{1}{s-r} + \log^+ \frac{1}{R-s} \\ &+ 3\log^+ T(R) + 4\log 2 + \frac{s-r}{R-s}\frac{R}{r} N_f(R, 0+\infty). \end{aligned}$$

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- For $r < R$ we choose s such that

$$\frac{s-r}{R-s}\frac{R}{r} = \frac{\frac{1}{2}}{T_f(R)+1}.$$

Then $\frac{s-r}{R-s}\frac{R}{r} N_f(R, 0+\infty) \leq 1$. Hence,

$$m_{f'/f}(r) \leq 3\log^+ R + 4\log^+ \frac{R}{r} + 5\log^+ \frac{1}{R-r} + 7\log^+ T(R) + O(1).$$

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- Remark: LDL can also be proved by the negative curvature method (G-J+Calculus Lemma) using

$$\partial_z \partial_{\bar{z}} (1 + (\log |f|^2)^2)^{1/2} = (1 + (\log |f|^2)^2)^{-3/2} |\partial_z \log f|^2$$
$$+ (1 + (\log |f|^2)^2)^{-1/2} (\log |f|^2) \partial_z \partial_{\bar{z}} \log |f|^2.$$

- Nevanlinna's SMT: Let $a_1, \ldots, a_q \in \mathbf{C} \cup \{\infty\}$. Let f be a non-constant meromorphic function on $\mathbf{C}$. Then for any $\epsilon > 0$, then $(q-2)T_f(r) \leq_{exc} \sum_{j=1}^{q} N_f^{(1)}(r, a_j) + O(\log rT_f(r))$.

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$$\log^+ |F(z)| \geq \sum_{j=1}^{q} \log^+ \frac{1}{|f(z) - a_j|} - q \log^+ \frac{2q}{\delta} - \log 2.$$

- Nevanlinna's SMT: Let $a_1, \ldots, a_q \in \mathbf{C} \cup \{\infty\}$. Let f be a non-constant meromorphic function on $\mathbf{C}$. Then for any $\epsilon > 0$, then $(q-2)T_f(r) \leq_{exc} \sum_{j=1}^{q} N_f^{(1)}(r, a_j) + O(\log rT_f(r))$.
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- Proof: Use Let $\delta = \min_{i \neq j}\{|a_i - a_j|, 1\}$. Consider $F(z) := \sum_{j=1}^{q} \frac{1}{f(z)-a_j}$. For each z, suppose that $|f(z) - a_{j_0}| < \delta/3q$, then, for $j \neq j_0$, $|f(z) - a_j| > \frac{2}{3}\delta$, so

  $$\log^+ |F(z)| \geq \sum_{j=1}^{q} \log^+ \frac{1}{|f(z) - a_j|} - q \log^+ \frac{2q}{\delta} - \log 2.$$

  The rest follows from the LDL by using $f'/(f-a_j) = (f-a_j)'/(f-a_j)$.

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The technique is called "pole-type converter" by Siu. We have, for $0 < r < R$ and $0 < \frac{n(n+1)t}{2} < p' < 1$,

$$\int_{|z|=r}\left|\frac{W(f_0,\ldots,f_n)}{L_1(f)\cdots L_q(f)}\right|^t \|f\|^{t(q-n-1)}\frac{d\theta}{2\pi} \leq C\left(\frac{R}{r(R-r)}T_f(R)\right)^{p'}.$$

- By Jensen's formula, we have

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- $H_1, \ldots, H_q$ (or $L_1, \ldots, L_q$) are said to be in n-subgeneral position if, for every $1 \leq i_0 < \cdots < i_n \leq q$, the linear forms $L_{i_0}, \ldots, L_{i_n}$ are linear independent.

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- If $H_1, \ldots, H_q$ in $\mathbf{P}^n(\mathbf{C})$ are in general position, then $H_1 \cap \mathbf{P}^m(\mathbf{C}), \ldots H_1 \cap \mathbf{P}^m(\mathbf{C})$ are in n-subgeneral position in $\mathbf{P}^m(\mathbf{C})$ for $n \geq m$.

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$$\prod_{j \in Y} E_j^{\omega(j)} \leq \prod_{j \in M} E_j.$$

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$$\sum_{j=1}^{q} m_{f,H_j}(r) + \left(\frac{n+1}{m+1}\right) N(R_f, r) \leq_{exc} (2n - m + 1) T_f(r)$$
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In particular, $\mathbf{P}^n$ minus $2n+1$ hyperplanes in general position is hyperbolic.

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where $\omega(j)$ is the Nochka weight corresponding to $\hat{H}_j$. Applying Nochka's theorem with $E_t = \frac{\|f(z)\| \|\hat{L}_{i(z,t)}\|}{|L_{i(z,t)}(f)(z)|} \geq 1, 0 \leq t \leq n$, there is a subset M of $Y = \{i(z,0), \ldots, i(z,n)\}$ with $\# M = m + 1$ such that $\{\hat{L}_{i(z,j)} | i(z,j) \in M\}$ is linearly independent, and

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Thus we can apply cartan's theorem.

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$$\sum_{j=1}^{q} d_j^{-1} m_f(r, D_j). \leq .(n+1+\epsilon) T_f(r).$$

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$$\int_0^{2\pi} \max_K \log \left(\prod_{j \in K} \frac{\|F(re^{i\theta})\| \|L_j\|}{|L_j(F)(re^{i\theta})|} \right) \frac{d\theta}{2\pi} . \leq .(n_m + 1 + \epsilon) T_F(r).$$

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- $S_X(m, \mathbf{c}) = \max(\sum_{j=1}^{n_m+1} \mathbf{a}_j \cdot \mathbf{c})$ for $\mathbf{c} = (c_0, \ldots, c_N)$ is called the Hilbert Weight of X with respect to $\mathbf{c}$.

- Theorem A: Assume that $f : \mathbf{C} \to X \subset \mathbf{P}^N$ is Zariski dense. For every $\epsilon > 0$, we have, for m big enough,

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  Theorem B together with the above result implies the following result:

- Theorem: Let $f = (f_0 : \cdots : f_N) : \mathbf{C} \to X \subset \mathbf{P}^N(\mathbf{C})$ be a Zariski-dense holomorphic map, where $\dim X = n$. Assume that the coordinate hyperplanes are in general position on X. Then, for every $\epsilon > 0$,

$$\int_0^{2\pi} \max_{i_0,\ldots,i_n} \log \prod_{k=0}^{n} \frac{\|f(re^{i\theta})\|}{|f_{i_k}(re^{i\theta})|} \frac{d\theta}{2\pi} \leq_{exc} (n+1+\epsilon) T_f(r).$$

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- Let $f : \mathbf{C} \to V$ be a Zariski-dense holomorphic map. For given hypersurfaces $D_1, \ldots, D_q$ defined by $Q_1, \ldots, Q_q$, assuming they have the same degree, we consider the map $\phi : V \to \mathbf{P}^{q-1}(\mathbf{C})$ defined by $\phi(w) = [Q_1(w) : \cdots : Q_q(w)]$. The above theorem, when applying to $X = \phi(V)$ and $F = \phi \circ f : \mathbf{C} \to X$, implies the SMT for hypersurfaces.

- Recall the definition of Chow weight: To $X \subset \mathbb{P}^N$ with $\dim X = n, \deg X = d$, we can associate, up to a constant scalar, a unique polynomial

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- Let t be an auxiliary variable. We consider the decomposition

$$F_X(t^{c_0}u_{00},\dots,t^{c_N}u_{0N};\dots;t^{c_0}u_{n0},\dots,t^{c_N}u_{nN})$$

$$= t^{e_0}G_0(u_0,\dots,u_n)+\cdots+t^{e_r}G_r(u_0,\dots,u_n)$$

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Hyperbolicity

- For a complex manifold, $n = dim_{\mathbf{C}} X$, one defines the Kobayashi pseudo-metric : $x \in X, \xi \in TX$

$$\kappa_x(\xi) = \inf\{\frac{1}{r}; \text{there is } f : D(r) \to X, f(0) = x, f_*(0) = \xi\}$$

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- For a complex manifold, $n = dim_{\mathbf{C}} X$, one defines the Kobayashi pseudo-metric : $x \in X, \xi \in TX$

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The negative curvature method

- Let X be a Riemann surface, i.e., a 1-dimensional complex manifold. Let
$$d\sigma^2 = 2a(z)dzd\bar{z}$$
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- Corollary: X be a Riemann surface with $K_{ds^2} \leq -1$. Then X is hyperbolic.

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The construction of metrics

- let $M = \mathbf{P}^1(\mathbf{C}) - \{a_i\}_{i=1}^q$ and let $\|z, a\|$ denote the spherical distance of $\mathbf{P}^1(\mathbf{C})$. Define a hermitian metric $d\sigma^2$ on M by

  $$d\sigma^2 = \frac{1}{\prod_{i=1}^q \|z, a_i\|^2 (\log c\|z, a_i\|^2)^2} \cdot \frac{4}{(1+|z|^2)^2} dz d\bar{z}$$

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- We take the geometric mean of σ_k and define $\Gamma = \frac{\sqrt{-1}}{2\pi} c \prod_{k=0}^{m-1} \sigma_k^{\beta_m/\lambda_k} dz \wedge d\bar{z}$, where $\beta_m = 1/\sum_{k=0}^{m-1} \lambda_k^{-1}$ and $c = 2(\prod_{k=0}^{m-1} \lambda_k^{\lambda_k^{-1}})^{\beta_m}$.

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- Write $\Gamma = \frac{\sqrt{-1}}{2\pi} h(z) dz \wedge d\bar{z}$, then $h(z) = c \prod_{j=1}^{q} \left(\frac{1}{\phi_0^{\omega(j)}(H_j)} \right)^{\beta_m} \prod_{j=1}^{q} \left[\prod_{k=0}^{m-1} \frac{a_k^{\beta_m/\lambda_k}}{(N - \log \phi_k(H_j))^{2\beta_m}} \right]$.

Extention of LDL

- Recall LDL: f meromorphic, then

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$$\begin{aligned}\omega \;\; = \;\; & \sum_{\nu_1+\cdots+\nu_n=k} a_{\nu_1,\ldots,\nu_n}(z_1,\ldots,z_n)\left(\frac{dz_1}{z_1}\right)^{\nu_1}\cdots\left(\frac{dz_q}{z_m}\right)^{\nu_m} \\ & \ldots (dz_{m+1})^{\nu_{m+1}}\cdots(dz_n)^{\nu_n}.\end{aligned}$$

- Generalized LDL: Let D be a divisor of simple normal crossing on a compact complex manifold M or let D be a trivial divisor. Let $\omega \in H^0(M, \Omega^k_{M,x}(\log D)$. Let $f : \mathbf{C} \to M$ be a holomorphic curve. Write $f^*\omega = \zeta dz$, then

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  Reason: Vanishing on an ample divisor E contributes to faster growth than $O(\log T_f(r) + \log r)$.

- One usually uses Riemann-Roch, together with some vanishing theorems, to produce a meromorphic differential $\omega \in H^0(M, \Omega^k_M(\log D) \otimes [-E])$ in order to use Schwarz lemma.

- The case of logarithmic jet differentials are defined analogously. Consider algebraic differential operators which can be written locally in multi-index notation

$$\begin{aligned} P(f_{[k]}) &= P(f', f'', ..., f^{(k)}) \\ &= \sum a_{\alpha_1 \cdots \alpha_k}(f(t)) f'(t)^{\alpha_1} f''(t)^{\alpha_2} \cdots f^{(k)}(t)^{\alpha_k} \end{aligned}$$

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  where $a_{\alpha_1 \cdots \alpha_k}(z)$ are holomorphic coefficients on X and $t \to z = f(t)$ is a curve, $f_{[k]} = (f', f'', ..., f^{(k)})$ its k-jet. Obvious $\mathbf{C}^*$-action :

  $$\lambda \cdot f(t) = f(\lambda t), \ \ (\lambda \cdot f)^{(k)}(t) = \lambda^k f^{(k)}(\lambda t)$$

  implies weighted degree $m = |\alpha_1| + 2|\alpha_2| + \cdots + k|\alpha_k|$.
- Define $E^{GG}_{k,m}$ = the sheaf (bundle) of algebraic differential operators of order k and weighted degree m.

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- By definition, proving the algebraic degeneracy means finding a nonzero polynomial P on M such that $P(f) = 0$.

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