MATH3338. TEST 1. SPRING 2013.
Write your name and UH-ID on the cover the blue book.
Start each problem on the new page. Clearly mark the answer to each problem.

1. (15pts) If \( A \) and \( B \) are events in the sample space \( S \), show \textbf{ONE} of the following set relations.

   (a) \( A \cup (A \cap B)^c = S \);
   
   (b) \( A = (A \cap B) \cup (A \cap B^c) \);
   
   (c) \( A \cup B = A \cup (A^c \cap B) \).

Write your explanations using English sentences. For example, for part c) you may start with: \textit{let \( x \) be a point in \( A \cup B \). It means, \( x \) is in \( A \) or \( x \) is in \( B \). If \( x \) is in \( A \), ... and so on. You may draw Venn diagrams to help your visualize the sets.}

2. (15pts) List the axioms of probability.

3. (15pts) A sample of 4 people is taken and every person is classified according to the type of his/her job (assume that it is either white-collar or blue-collar) and the political affiliation (assume that it is either Republican, Democrat or Independent).
   
   (a) How would you set up the sample space (set of outcomes) for this experiment?
   
   (b) How many outcomes are in the sample space?
   
   (c) Using the your notation from part a), list all outcomes in the event that neither of the people is Republican or Democrat.

4. (20pts) An sample of 100 high school students is taken. It is observed that 28 students take Chemistry class, 26 Physics class and 16 take Math. There are 12 students that take both Chemistry and Physics, 4 take Chemistry and Math., and 6 that take both Physics and Math. In addition, there are 2 students taking all three classes.
   
   (a) If a student is chosen at random, what is the probability that he/she is not in any of these classes?
   
   (b) What is the probability that he/she is taking exactly one of the Chemistry, Physics or Math. classes?

5. (15pts) There are 50 hotels in a city. 10 people are to check into hotels. Assuming that all hotels are equally likely to be chosen by a person, what is the probability that at least two check into the same hotel.

6. (20pts) In a community, 36% of the families own a dog, and 22% of the families that own a dog also own a cat. In addition, 30% of the families own a cat.
   
   (a) What is the probability that a randomly selected family owns both a dog and a cat?
   
   (b) What is the conditional probability that a randomly selected family owns a dog, given that it owns a cat?

\textbf{extra cr.} (5pts): In problem 4, if two students are chosen randomly, what is the probability that at least one is taking one of the Chemistry, Physics or Math. classes?
TEST 1. Solutions

1. a) $A \cup (A \cap B)^c = S$

Since $A \cap B$ subsets in the sample space $S$, $A \cup (A \cap B)^c$ is contained in $S$.

Let $x \in S$. If $x \in A$, then

$x \in A \cup (A \cap B)^c$.

If $x \notin A = \Rightarrow x \in (A \cap B)^c = \Rightarrow x \in (A \cap B)^c$.

It follows that $S$ is contained in $A \cup (A \cap B)^c$.

b) $A = (A \cap B) \cup (A \cap B)^c$

Let $x \in (A \cap B) \cup (A \cap B)^c$

$\Rightarrow x \in A \cap B$ or $x \in A \cap B^c$

If $x \in A \cap B$ $\Rightarrow x \in A$ and $x \in B$

If $x \in A \cap B^c$ $\Rightarrow x \in A$ and $x \in B^c$

In both cases $x \in A \Rightarrow$ Set on the right is contained in the set on the left.

Let $x \in A$. If $x \in B = \Rightarrow x \in A \cap B$

If $x \in B^c = \Rightarrow x \in A \cap B^c$

$\Rightarrow x \in (A \cap B) \cup (A \cap B)^c$. 
\[ \Rightarrow \text{ } A \text{ is contained in } (A \cap B) \cup (A \cap B^c) \]

C) \[ A \cup B = A \cup (A^c \cap B) \]

Let \( x \in A \cup B \Rightarrow x \in A \text{ or } x \in B. \)

If \( x \in A, \) then also \( x \in A \cup (A^c \cap B) \)

If \( x \in B, \) and \( x \in A^c, \) then \( x \in A \cup (A^c \cap B). \)

\[ \Rightarrow \text{ } A \cup B \text{ is contained in } A \cup (A^c \cap B). \]

Let \( x \in A \cup (A^c \cap B) \)

\[ \Rightarrow x \in A \text{ or } (x \in A^c \text{ and } x \in B) \]

\[ \begin{align*}
\text{In either case } x & \in A \cup B.
\end{align*} \]

\[ \text{#2} \]

Axioms:

1. \( p(A) \geq 0 \), for any event \( A. \)

2. \( p(S) = 1 \)

3. \( p(\bigcup_{i=1}^{N} A_i) = \sum_{i=1}^{N} p(A_i) \) if \( A_k \cap A_j = \emptyset \)

\[ \text{for any } k \neq j. \]
3a) Each person can be assigned a pair of letters \((a, b)\), where \(a \in \{w, b\}\) and \(b \in \{R, D, I\}\).

Sample space consists of all combinations of

\[
S = \left\{ \left( (a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4) \right) : a_i \in \{w, b\}, b_i \in \{R, D, I\} \right\}.
\]

b) There are 6 pairs \((a, b)\), and

\[S\] contains \(4 \times 6 = 24\) points.

c) \[
\left( (w, I), (w, I), (w, I), (w, I) \right), \\
\left( (b, I), (w, I), (w, I), (w, I) \right), \\
\left( (w, I), (b, I), (w, I), (w, I) \right), \\
\left( (w, I), (w, I), (b, I), (w, I) \right), \\
\left( (w, I), (w, I), (w, I), (b, I) \right), \\
\left( (b, I), (b, I), (w, I), (w, I) \right), \\
\ldots \text{ and so on.} \quad 16\text{ points.}
\]
Let $x, y, z, w$ be the number of people in corresponding sets.

$W = 2$
$W + Y = 12 \Rightarrow Y = 10$
$W + z = 6 \Rightarrow z = 4$
$W + X = 4 \Rightarrow X = 2$

Number of people taking either Ch., Ph., or Math:

$$= 28 + 26 + 16 - 12 - 4 - 6 + 2$$

$$= 50$$

Number of people not taken Ch., Ph., or Math:

$$= 100 - 50 = 50$$

a) $P(\text{student not taken any of 3 classes}) = \frac{50}{100} = \frac{1}{2}$. 

b) $P(\text{student takes exactly one class}) = \frac{14 + 10 + 10}{100}$

$$= \frac{34}{100} = \frac{34}{100}$$
The problem can also be solved using rules of probability.

Let $A$ - event that a randomly chosen student takes Chem.

$B \quad \text{Ph:}$

$C \quad \text{Math:}$

$p(A) = \frac{28}{100}$, $p(B) = \frac{26}{100}$, $p(C) = \frac{16}{100}$

$p(ANB) = \frac{12}{100}$, $p(ANC) = \frac{4}{100}$, $p(BNC) = \frac{6}{100}$

$p(ANBNC) = \frac{2}{100}$.

(a) Need $p\left( (A \cup B \cup C) \right) = 1 - p(\overline{A \cup B \cup C}) =$

$= 1 - \left( p(A) + p(B) + p(C) - p(ANB) - p(ANC) - p(BNC) + p(ANBNC) \right)$

$= 1 - \frac{28 + 26 + 16 - 12 - 4 - 6 + 2}{100} = \frac{56}{100}$

(b) Need $p\left( (A \setminus (B \cup C)) \cup (B \setminus (A \cup C)) \cup (C \setminus (A \cup B)) \right)$

$p\left( (A \setminus (B \cup C)) \cup (B \setminus (A \cup C)) \cup (C \setminus (A \cup B)) \right)$

$= p(A) - p(ANB) + p(ANC) - p(ANBNC) + p(B) - p(ANB) + p(BNC) - p(ANBNC) + p(C) - p(ANC) + p(BNC) - p(ANBNC)$

$= \frac{52}{100}$.
#5. \[ p( \text{at least two into the same hotel}) = 1 - p(\text{all into different}) \].

\( (50)^{10} \) - arrangements of hotels for 10 people.

\( 50 \cdot 49 \cdots 41 \) - arrangements of different hotels for 10 people.

\[ \Rightarrow p = 1 - \frac{50 \cdot 49 \cdot 48 \cdots 41}{(50)^{10}}. \]

#6. Let \( A \) - event that a randomly chosen family has a dog.

\( B \) - has a cat.

\( P(A) = \frac{36}{100}, \quad P(B|A) = \frac{22}{100}, \quad P(B) = \frac{30}{100} \)

\( \circ \quad P(\text{A} \cap \text{B}) = P(B|A) \cdot P(A) = \frac{36 \cdot 22}{100^2} \)

\( \circ \quad P(\text{A} | \text{B}) = \frac{P(\text{A} \cap \text{B})}{P(B)} = \frac{36 \cdot 22}{100^2} \cdot \frac{100}{30} = \frac{36 \cdot 22}{30 \cdot 100} \).
Extra credit

\[ p = 1 - p (\text{none of two is taking Chem., Ph. or Math}) \]

There are 100.99 ways to choose two people out of 100 in the ordered way.

Since 50 students do not take Chem., Ph. or Math, the first student is any out of 50, and 2nd is any out of 49:

\[ p = 1 - \frac{50 \cdot 49}{100 \cdot 99} . \]

You can also do it in unordered way:

\[ p = 1 - \binom{50}{2} \binom{100}{2} . \]