Approximate Solutions of Scalar Conservation Laws

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Abstract. We study compactness properties of time-discrete and continuous time BGK-type schemes for scalar conservation laws, in which microscopic interactions occur only when the state of a system deviates significantly from an equilibrium distribution. The threshold deviation, \( \varepsilon \), is a parameter of the problem. In the vanishing relaxation time limit we obtain solutions of a conservation law in which flux is pointwisely close (of order \( \varepsilon \)) to the flux of the original equation and derive several other properties of such solutions, including an example of approximate solution to a shock for Burger's equation.

Key words. Scalar conservation laws, kinetic models, shock waves.

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1. Introduction

Several important classes of PDEs can be formally represented by using kinetic formalism by extending the space of independent variables. In fact, many PDEs, such as the Euler equations of gas dynamics can be derived from the kinetic equations of microscopic description of motion. A kinetic representation of equations of the type

\[ \partial_t U + \text{div}_x F(U) = 0, \]  

(1.1)

where \((x,t) \in \mathbb{R}^{d+1}, U : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d, \) and \( F(U) : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}, \) uses a microscopic variable \( v \) and a set of equilibria \( M_0 \) that consist of functions \( f(U,v) \) such that for some functions \( \kappa(v) \in \mathbb{R}^d \) and \( a(v) \in \mathbb{R}^d, \) such that for any \( U \in \mathbb{R}^d, \)

\[ U = \int \kappa(v)f(U,v)dv, \quad F(U) = \int \kappa(v) \otimes a(v)f(U,v)dv. \]

One attempts to solve the kinetic equation for unknowns \((f,Q)\):

\[ \partial_t f + a(v) \cdot \nabla_x f = Q, \]  

(1.2)

such that for all \((x,t)\)

\[ f(x,t,\cdot) \in M_0, \quad \int \kappa(v)Q(x,t,v)dv = 0. \]

Often more information is available: the set of equilibria \( M_0 \) can be characterized as a set of constrained minimizers of an entropy (convex function) \( S(f) :\)

\[ \min\{S(f) : \int \kappa(v)f(v)dv = U = \text{const.}\}. \]

For such \( f, \int S(f(U,v))dv = s(U) \) is an entropy function for (1.1):

\[ \partial_t s + \text{div}_x q \leq 0, \]

where \( q = q(U) \) is the entropy flux. We will use notation \( \Pi f \in M_0 \) for the above minimizer with \( \kappa \)-moments:

\[ \int \kappa(v)\Pi f(v)dv = \int \kappa(v)f(v)dv. \]