1. Introduction

In this paper we consider the problem of obtaining solutions of the Euler equations for

\[
\begin{align*}
\rho_t + (m)_r + \frac{nm}{r} &= 0, \\
(m)_t + \left(\frac{m^2}{\rho} + p(\rho)\right)_r + \frac{nm^2}{\rho r} &= 0,
\end{align*}
\]

in the limit from a sequence of vanishing viscosity solutions of the Euler equations with viscosity,

\[
\begin{align*}
\rho_t + (m)_r + \frac{nm}{r} &= \varepsilon \left(\rho_{rr} + \frac{nm}{r}\right), \\
(m)_t + \left(\frac{m^2}{\rho} + p(\rho)\right)_r + \frac{nm^2}{\rho r} &= \varepsilon \left((m)_r + \frac{nm}{r}\right),
\end{align*}
\]

In the above equations \(\rho, u\) is the radial density and radial velocity of the gas, pressure \(p(\rho) = \rho^\gamma\) and \(n = N - 1\), \(N > 1\) is the dimension of the space. We consider (1.2) on the cylinder \(Q = [0, +\infty) \times \Omega, \Omega = [a, b]\), with \(a, b \in (0, +\infty)\), and the boundary conditions

\[
m(t, r)|_{r=a,b} = 0, \quad \rho_r(t, a) = 0, \quad \rho(t, b) = \bar{\rho}, \quad t > 0,
\]

for some \(\bar{\rho} > 0\) and initial conditions

\[
(\rho, m)(t = 0) = (\rho_0(x), m_0(x)).
\]

Let's state the main result of this paper. Let \((\rho_0, m_0) \in L^1_{loc}((0, +\infty))^2\), with \(\rho_0 \geq 0\), has a finite total energy:

\[
\left(\frac{m_0^2}{2\rho_0} + \frac{\rho_0^\gamma}{\gamma - 1}\right) r^n \in L^1((0, +\infty)),
\]

Let \(a = a(\varepsilon)\) with \(\lim a(\varepsilon) = 0\), \(b = b(\varepsilon)\) with \(\lim b(\varepsilon) = 0\), as \(\varepsilon \to 0\) and denote \(\Omega = (a, b)\). We'll restrict \(\gamma \in (1, 2]\).

**Theorem 1.1.** There is \(\bar{\rho} = \bar{\rho}(\varepsilon)\), with \(\lim \bar{\rho}(\varepsilon) = 0\), such that if \((\rho_0^\varepsilon, m_0^\varepsilon)\), is a sequence of smooth functions with properties

i. \((\rho_0^\varepsilon, m_0^\varepsilon)\) verifies assumptions of theorem 2.1;

ii. \((\rho_0^\varepsilon, m_0^\varepsilon) \to (\rho_0, m_0)\) a.e. \(r \in (0, +\infty)\), where by \((\rho_0^\varepsilon, m_0^\varepsilon)\) we understand the zero extension of \((\rho_0^\varepsilon, m_0^\varepsilon)\) outside \(\Omega\);
iii \( \int_{\Omega} (m_0^\varepsilon)^2 / \rho_0^\varepsilon + (\rho_0^\varepsilon)^\gamma r^n dr \rightarrow \int_{\Omega} (m_0^2 / \rho_0 + (\rho_0)^\gamma) r^n dr \), where \( E \) is defined in (2.1);

and \((\rho^\varepsilon(t,r), \psi^\varepsilon(t,r))\) - the unique classical solution of (1.2)–(1.4) with initial data \((\rho_0^\varepsilon, \psi_0^\varepsilon)\), then there is a subsequence, still labeled \((\rho^\varepsilon, \psi^\varepsilon)\), that converges a.e. \((t,r) \in Q\) and in \(L^p_{loc}(Q) \times L^q_{loc}(Q), p \in [1, \gamma + 1], q \in [1, 3(\gamma + 1)/(\gamma + 3)]\), to \((\rho, m)\) - a weak energy solution of the Euler equations in the following sense: \(\forall \omega \in C_0^\infty([0, +\infty) \times [0, +\infty))\), with \(\omega_r(t,0) = 0\),

\[
\int_Q \rho \omega_t + m \omega_r (r^n dr) dt + \int_{\Omega} \rho_0 \omega(0,r) r^n dr = 0,
\]

for all \(\omega \in C_0^\infty([0, +\infty) \times [0, +\infty))\), with \(\omega(t,0) = \omega_r(t,0) = 0\),

\[
\int_Q m \omega_t + (m^2 / \rho + \rho^\gamma) \omega_r (r^n dr) dt + \int_{\Omega} m_0 \omega(0,r) dr = 0,
\]

a.e. \(t\),

\[
\int_{(0, +\infty)} \left( \frac{m^2}{\rho} + \rho^\gamma \right)(t,r) r^n dr \leq \int_{(0, +\infty)} \left( \frac{m_0^2}{\rho_0} + \rho_0^\gamma \right)(r) r^n dr,
\]

(1.5)

for any \(\psi(s)\) - convex function with subquadratic growth at infinity and \((\eta_\psi, q_\psi)\) - entropy/entropy flux pairs defined in (3.13), (3.14),

\[
(\eta_\psi r^n)_t + (q_\psi r^n)_r + mn^{n-1} (m\eta_\psi, \rho + \frac{m^2}{\rho} \eta_\psi, m - q_\psi) \leq 0, \quad \mathcal{D}'((0, +\infty) \times (0, +\infty)).
\]

2. Global existence of a unique classical solution to the artificial viscosity equations

Equations (1.2) form a parabolic, quasi-linear system of equations in \((\rho, m)\). In this section we will prove existence of a unique solution \((\rho, u)\) in the classes of smooth functions. We prove the following existence result under the assumption \(\gamma \in (1, 2]\). For \(\beta \in (0, 1)\) let \(H^{2+\beta}(\Omega)\) and \(H^{2+\beta,1+2/\gamma}(Q_T)\) be the usual Hölder and parabolic Hölder spaces. Their definitions can be found for example in [3]. We take \((\rho_0, m_0) \in (H^{2+\beta}(\Omega))^2\) with \(\inf_{\Omega} \rho_0 > 0\). The following theorem holds.

**Theorem 2.1.** Let, in addition to conditions stated above, assume that the following compatibility conditions hold:

\[
\rho_0(a) = 0, \quad \rho_0(b) = \bar{\rho}, \quad m_0(r)|_{r=a,b} = 0,
\]

at \(r = a\),

\[
\left( m_{0,r} + \frac{nm_0}{r} \right)_r = 0,
\]

at \(r = b\),

\[
m_0 = \varepsilon \left( \rho_{0,rr} + \frac{n\rho_0}{r} \right), \quad \left( \frac{m_0^2}{\rho_0} + p(\rho_0) \right)_r = \varepsilon \left( m_{0,r} + \frac{nm_0}{r} \right)_r.
\]
Then, there is a unique solution \((\rho, m)\) of the problem (1.2)–(1.4) such that \((\rho, m) \in (H^{2+\beta,1+\frac{\beta}{2}}(\Omega_T))^2\), and \(\inf_{Q_T} \rho > 0\), \(\forall T > 0\).

**Proof.** The nonlinear terms in (1.2) have singularities at \(\rho = 0\) and \(\rho, |m| = +\infty\). To prove the theorem we will derive a priori estimates for a generic \(C^{2,1}(\Omega_T)\) solution in the norms \(\|\rho\|_{L^\infty(\Omega_T)}, \|\frac{1}{\rho}\|_{L^\infty(\Omega_T)},\) and \(\|\frac{m}{\rho}\|_{L^\infty(\Omega_T)}\), showing by this that the solution takes values in a region (determined a priori) away from singularities. The existence of the solution stated in the theorem can be derived from the general theory of the parabolic quasi-linear systems, by a suitable linearization techniques, see §5 and Theorem 7.1 of [3].

A priori estimate are obtained by the follow arguments: first we derive the estimates based on the balance of total energy. In lemma 2.1 we use the maximum principle for the Riemann invariants and the total energy estimates to show that the \(L^\infty\) norm of \(u\) depends linearly on \(L^\infty\) norm of \(\rho\), with \(\theta = \gamma - 1\). This in turn we use in lemma 2.2 to close higher energy estimates for \(\rho\) and \(m\). With that we obtain a priori bound of \(L^\infty\) norm of \(\rho\), and by using again lemma 2.1, of \(L^\infty\) norms of \(u\) and \(m\). Finally, to show positive lower bound for \(\rho\) we obtain an estimate on \(\int_0^t \|u_r\|_{L^\infty} dt\).

We proceed now with the derivation of a priori estimates. Let \((\rho, m)\), with \(\rho > 0\) be a \(C^{2,1}(\Omega_T)\) solution of (1.2)–(1.4).

As usual we denote by

\[
\eta^* = \frac{m^2}{2\rho} + e(\rho), \quad q^* = \frac{m^3}{2\rho^2} + me'(\rho), \quad e(\rho) = \rho^\gamma/\gamma - 1,
\]

and by

\[
E(\rho, m) = \eta^*(\rho, m) - \eta^*(\bar{\rho}, 0) - \eta^*_\rho(\bar{\rho}, 0)\rho - \rho^\theta/\gamma - 1
\]

– the total energy relative to the state \((\bar{\rho}, 0)\). Note that \((\bar{\rho}, 0)\) is the only equilibrium solution of the system. We multiply the first equation in (1.2) by \(E_\rho = (\eta^* - \eta^*_\rho(\bar{\rho}, 0))r^n\), the second by \(E_m = \eta^*_m r^n\), add them up to get

\[
(E_r^n)_t + (q^* r^n - \eta^*_\rho(\bar{\rho}, 0)mr^n)_r = \varepsilon r^n \left(\rho_{rr} + \frac{n\rho_r}{r}\right) (\eta^*_\rho - \eta^*_\rho(\bar{\rho}, 0)) + \varepsilon r^n \left(m_{rr} + \frac{nm}{r}\right) \eta^*_m,
\]

and

\[
(E_r^n)_t + (q^* r^n - \eta^*_\rho(\bar{\rho}, 0)mr^n)_r + n\varepsilon m\eta^*_m r^{n-2} = \varepsilon (\rho_r r^n)_r (\eta^*_\rho - \eta^*_\rho(\bar{\rho}, 0)) + \varepsilon (m_r r^n)_r \eta^*_m.
\]
Integrating over $Q_T$, using boundary conditions (1.3) we get:

$$\sup_{t \in [0,T]} \int_{\Omega} E(\rho(t, r), m(t, r)) r^n \, dr$$

$$+ \varepsilon \int_{Q_T} \left\{ \langle \nabla^2 E(\rho, m)(\rho_r, m_r), (\rho_r, m_r)^T \rangle + \frac{m^2}{2 \rho r^2} \right\} r^n \, dr \, dt = E_0. \quad (2.3)$$

Note, that $\langle \nabla^2 E(\rho, m)(\rho_r, m_r), (\rho_r, m_r)^T \rangle$ is a positive definite form that dominates $\rho^{\gamma - 2}\rho_r^2$ and $\rho |u_r|^2$, so that

$$\varepsilon \int_{Q_T} (\rho^{\gamma - 2}\rho_r^2 + \rho |u_r|^2) r^n \, dr \, dt \leq E_0. \quad (2.4)$$

Estimate (2.3) implies:

$$\sup_{t \in [0,T]} \int_{\Omega} [e(\rho(t, \cdot)) - e(\bar{\rho}) - e'(\bar{\rho})(\rho(t, \cdot) - \bar{\rho})] r^n \, dr \leq E_0. \quad (2.5)$$

Function $G(\rho) = e(\rho) - e(\bar{\rho}) - e'(\bar{\rho})(\rho - \bar{\rho})$ is positive, quadratic in $\rho - \bar{\rho}$ for $\rho$ near $\bar{\rho}$, and growing as $\rho^2$ for large values of $\rho$. Thus, for any $t \in [0, T]$ the measure of the set $\{\rho(t, \cdot) > \frac{3}{2}\rho\}$ is less than $c(\bar{\rho})E_0$ for certain $c(\bar{\rho}) > 0$. Let $r \in \Omega$ and $r_0 \in \Omega$ is the closest to $r$ point such that $\rho(t, r_0) = \frac{3}{2}\bar{\rho}$. Clearly, $|r - r_0| \leq c(\rho)E_0$. With such choice of $r_0$ we estimate (Int$[a, b]$ – the interval with endpoints $a, b$):

$$|\rho^\gamma(t, r) - \rho^\gamma(t, r_0)| \leq \gamma \int_{\text{Int}[r, r_0]} \rho^{\gamma - 1}-1|\rho_r|^2 \, dr$$

$$\leq c(a, \bar{\rho}) \left( \int_{\text{Int}[r, r_0]} \rho^\gamma(t, r) r^n \, dr \right)^{\frac{1}{2}} \left( \int_{\Omega} \rho^{\gamma - 2}|\rho_r|^2 r^n \, dr \right)^{\frac{1}{2}}$$

$$\leq c(a, \bar{\rho}, E_0) \left( \int_{\Omega} \rho^{\gamma - 2}(t, r)|\rho_r(t, r)|^2 r^n \, dr \right)^{\frac{1}{2}}.$$ 

Because of (2.3) we conclude that

$$\int_0^T \|\rho^\gamma(t, \cdot)\|_{L^\infty}^2 \, dt \leq c(a, \bar{\rho}, E_0, T). \quad (2.6)$$

2.1. Maximum principle estimates.

**Lemma 2.1.** There is $c = c(a, T, E_0)$ such that for any $t \in [0, T]$,

$$\|u\|_{L^\infty(Q_t)} \leq c(\|u_0 + \rho_0^{\theta}\|_{L^\infty(\Omega)} + \|u_0 - \rho_0^{\theta}\|_{L^\infty(\Omega)} + \|\rho\|_{L^\infty(\Omega)}). \quad (2.7)$$

**Proof.** Consider the Riemann invariants $w_1 = \frac{m}{\rho} + \rho^{\theta}$, $w_2 = \frac{m}{\rho} - \rho^{\theta}$. We will use the following notation $w_{1, \rho} = \partial_\rho w_1(\rho, m)$, $w_{1, m} = \partial_m w_1(\rho, m)$, $w_{1, r} = \partial_r(w_1(\rho(t, r), m(t, r)))$, $w_{1, t} = \partial_t(w_1(\rho(t, r), m(t, r)))$ and so on.
We multiply the first equation in (1.2) by $w_{1,r}$, the second equation by $w_{1,m}$ and add them to obtain

$$w_{1,t} + \lambda_{1}w_{1,r} + \frac{n\theta\rho^{\theta}u}{r} = -\varepsilon(\rho_{r}(w_{1,\rho})_{r} + m_{r}(w_{1,m})_{r}) + \varepsilon w_{1,rr} + \frac{n\varepsilon}{r} \left[ w_{1,r} - \frac{1}{r} mw_{1,m} \right],$$

where $\lambda_{1}$ is the corresponding eigenvalue. Thus, we obtain

$$w_{1,t} + \left( \lambda_{1} - \frac{n\varepsilon}{r} \right)w_{1,r} - \varepsilon w_{1,rr} = -\varepsilon \langle \nabla^{2}w_{1}(\rho_{r},m_{r})^{T},(\rho_{r},m_{r})^{T} \rangle - \frac{n\theta\rho^{\theta}u}{r} - \frac{n\varepsilon u}{r^{2}},$$

where by $\nabla^{2}w_{1}$ we mean the matrix of second derivatives of $w_{1}$ w.r.t. $(\rho,m).$ We write $(\rho_{r},m_{r}) = \alpha \nabla w_{1}(\rho,m) + \beta \nabla_{\bot} w_{1}(\rho,m),$ where

$$\alpha = \frac{w_{1,r}}{|\nabla w_{1}|^{2}}, \quad \beta = \frac{\rho_{r}w_{1,m} - m_{r}w_{1,\rho}}{|\nabla w_{1}|^{2}}.$$

We can write further

$$w_{1,t} + \lambda w_{1,r} - \varepsilon w_{1,rr} = -\varepsilon \beta^{2} \langle \nabla^{2}w_{1}(\nabla_{\bot} w_{1})^{T},(\nabla_{\bot} w_{1})^{T} \rangle - \frac{n\theta\rho^{\theta}u}{r} - \frac{n\varepsilon u}{r^{2}},$$

(2.8)

$$\lambda = \lambda_{1} - \frac{2\varepsilon}{r} + \varepsilon \alpha \langle \nabla^{2}w_{1}(\nabla w_{1})^{T},(\nabla w_{1})^{T} \rangle + 2\beta \langle \nabla^{2}w_{1}(\nabla_{\bot} w_{1})^{T},(\nabla w_{1})^{T} \rangle.$$ 

However $\langle \nabla^{2}w_{1}(\nabla_{\bot} w_{1})^{T},(\nabla_{\bot} w_{1})^{T} \rangle \geq 0$ and by setting

$$\bar{w}_{1}(t,r) = w_{1}(t,r) - \int_{0}^{t} \left| \frac{n\theta\rho^{\theta}u}{r} + \frac{n\varepsilon u}{r^{2}} \right|_{L^{\infty}(a,b)} d\tau,$$

we get by the classical maximum principle applied to the parabolic equation (2.8),

$$\max_{[0,t] \times \Omega} \bar{w}_{1} \leq \max \left\{ \max_{\Omega} w_{1,0}, \max_{[0,t] \times \Omega} \bar{w} \right\},$$

or

$$\max_{[0,t] \times \Omega} w_{1} \leq \max_{\Omega} w_{1,0} + \|\rho\|_{L^{\infty}(Q_{t})}^{\theta} + c(\bar{\rho},a) \int_{0}^{t} (1 + \|\rho(\tau,\cdot)\|_{L^{\infty}(\Omega)}^{\theta}) \|u(\tau,\cdot)\|_{L^{\infty}(\Omega)} d\tau.$$
In a similar way,
\[
\max_{[0,t] \times \Omega} (-w_2) \leq \max_{\Omega} (-w_{2,0}) + \|\rho\|^\theta_{L^\infty(Q_t)} + c(\bar{\rho}, a) \int_0^t (1 + \|\rho(\tau, \cdot)\|^\theta_{L^\infty(\Omega)}) \|u(\tau, \cdot)\|_{L^\infty(\Omega)} \, d\tau.
\]
Since \( \rho \geq 0 \), it follows that
\[
\max_{[0,t] \times \Omega} |u| \leq \max_{\Omega} |w_{1,0}| + \max_{\Omega} |w_{2,0}| + \max_{\Omega} \|\rho\|^\theta_{L^\infty(Q_t)} + c (\bar{\rho}, a) \int_0^t (1 + \|\rho(\tau, \cdot)\|^\theta_{L^\infty(\Omega)}) \|u(\tau, \cdot)\|_{L^\infty(\Omega)} \, d\tau. \tag{2.9}
\]
However \( \int_0^T \|\rho(\tau, \cdot)\|^\theta_{L^\infty} \, d\tau = \int_0^T \|\rho(\tau, \cdot)\|^\frac{\theta}{2} \, d\tau \leq C(a, \bar{\rho}, T, E_0) \), by (2.6), and we conclude (2.7) from (2.9).

2.2. Lower bound on \( \rho \).

**Lemma 2.2.** Let \( \gamma \in (1, 2] \). It holds:
\[
\sup_{t \in [0,T]} \int_{\Omega} \left[ |\rho_r|^2 + |m_r|^2 \right] \, dr + \int_0^T \int_{\Omega} \left[ |\rho_{rr}|^2 + |m_{rr}|^2 \right] \, dr\, dt 
\leq C(\varepsilon, a, \bar{\rho}, T, \|\rho_0\|_{H^2(\Omega)}, \|u_0\|_{L^\infty(\Omega)}, \|\rho_0\|_{L^\infty(\omega)}). \tag{2.10}
\]

**Proof.** We multiply the first equation in (1.2) by \( \rho_{rr} \), the second by \( m_{rr} \) to obtain
\[
\begin{align*}
-\partial_t \left[ \frac{|\rho_r|^2 + |m_r|^2}{2} \right] &- \varepsilon(|\rho_{rr}|^2 + |m_{rr}|^2) + (\rho_t \rho_r)_r + (m_t m_r)_r = \\
&= -m_r \rho_{rr} - \frac{nm}{r} \rho_{rr} - (\rho u^2 + p)_r m_{rr} - \frac{n\rho u^2}{r} m_{rr} \\
&+ \frac{n \varepsilon \rho_r \rho_{rr}}{r} + \left( \frac{n \varepsilon m}{r} \right)_r m_{rr}.
\end{align*}
\]
We integrate this equation over \( Q_t \):
\[
\begin{align*}
\int_{\Omega} \left[ \frac{|\rho_r|^2 + |m_r|^2}{2} \right] \, dr &+ \varepsilon \int_{Q_t} (|\rho_{rr}|^2 + |m_{rr}|^2) \, dr\, dt \\
&= \int_{Q_t} m_r \rho_{rr} \frac{nm}{r} \rho_{rr} \, dr\, dt + \int_{Q_t} (\rho u^2 + p)_r m_{rr} \, dr\, dt \\
&+ \int_{Q_t} \left( \frac{n \varepsilon \rho_r \rho_{rr}}{r} - \frac{n \varepsilon m}{r} \right)_r m_{rr} \, dr\, dt \\
&- \int_{Q_t} \left( \frac{n \varepsilon m}{r} \right)_r m_{rr} \, dr\, dt. \tag{2.11}
\end{align*}
\]
Let us estimate term \( \int_{Q_T} (\rho u^2 + p) m_{rr} \, dr \, dt \) first. Consider

\[
\left| \int_{Q_t} \rho^{\gamma-1} \rho_r m_{rr} \, dr \, d\tau \right| \leq \delta \int_{Q_t} |m_{rr}|^2 \, dr \, d\tau + C_\delta \int_{Q_t} \rho^{2\gamma-2} |\rho_r|^2 \, dr \, d\tau
\]

\[
\leq \delta \int_{Q_t} |m_{rr}|^2 \, dr \, d\tau + C_\delta \int_0^t \left( \|\rho(\tau, \cdot)\|_{L^\infty} \int_{\Omega} \rho^{\gamma-2} |\rho_r|^2 \, dr \right) d\tau.
\]

Using estimate (2.6) we can write for \( t \in [0, T] \),

\[
\|\rho(t, \cdot)\|_{L^\infty} \leq c(a, \bar{\rho}, E_0)(1 + \int_{\Omega} |\rho_r(t, \cdot)|^2 \, dr).
\]

Using this in the previous estimate we obtain

\[
\left| \int_{Q_t} \rho^{\gamma-1} \rho_r m_{rr} \, dr \, d\tau \right| \leq \delta \int_{Q_t} |m_{rr}|^2 \, dr \, d\tau
\]

\[
+ C_\delta \int_0^t (1 + \int_{\Omega} |\rho_r(\tau, \cdot)|^2 \, dr) \int_{\Omega} \rho^{\gamma-2} |\rho_r(\tau, r)|^2 \, dr \, d\tau.
\]

Consider \( (\rho u^2)_r m_{rr} = u^2 \rho_r m_{rr} + 2\rho uu_r m_{rr} \), we can estimate

\[
\int_{Q_t} |u^2 \rho_r m_{rr}| \, dr \, d\tau \leq \delta \int_{Q_t} |m_{rr}|^2 \, dr \, d\tau
\]

\[
+ C_\delta \int_0^t \|u(\tau, \cdot)\|_{L^\infty}^4 \int_{\Omega} |\rho_r(\tau, r)|^2 \, dr \, d\tau
\]

\[
\leq \delta \int_{Q_t} |m_{rr}|^2 \, dr \, d\tau
\]

\[
+ C_\delta \int_0^t \|u(\tau, \cdot)\|_{L^\infty}^4 \|\rho(\tau, \cdot)\|_{L^\infty}^2 \int_{\Omega} \rho^{\gamma-2} \rho_r(\tau, r)|\rho_r(\tau, r)|^2 \, dr \, d\tau,
\]

where in the last inequality we used the fact that \( \gamma \in (1, 2] \). Using the uniform estimates of lemma 2.1 we obtain

\[
\|u(\tau, \cdot)\|_{L^\infty}^4 \leq \sup_{[0, \tau] \times \Omega} |u|^4 \leq c(\bar{\rho}, a, \|u_0, \rho_0\|_{L^\infty})(1 + \sup_{[0, \tau] \times \Omega} \rho^{2\gamma-2}).
\]

Inserting this into the above inequality we get

\[
\int_{Q_t} |u^2 \rho_r m_{rr}| \, dr \, d\tau \leq \delta \int_{Q_t} |m_{rr}|^2 \, dr \, d\tau
\]

\[
+ C_\delta \int_0^t \left( 1 + \sup_{s \in [0, \tau]} \|\rho(s, \cdot)\|_{L^\infty}^2 \right) \int_{\Omega} \rho^{\gamma-2} \rho_r(\tau, r)|\rho_r(\tau, r)|^2 \, dr \, d\tau.
\]
Using (2.12) we get
\[ \int_{Q_t} u^2 \rho_r m_r \, dr \, d\tau \leq \delta \int_{Q_t} |m_r|^2 \, dr \, d\tau + C_\delta \int_0^t \left( 1 + \sup_{s \in [0, \tau]} \int_\Omega |\rho_r(s, r)|^2 \, dr \right) \left( \int_\Omega \rho_{\gamma - 2}(\tau, r)|\rho_r(\tau, r)|^2 \, dr \right) d\tau. \]  

(2.14)

Lastly,
\[ \int_{Q_t} |\rho u u_r m_{rr}| \, dr \, d\tau \leq \delta \int_{Q_t} |m_r|^2 \, dr \, d\tau + C_\delta \int_0^t \left( \int_\Omega |\rho_r(\tau, \cdot)u^2(\tau, \cdot)|_{L^\infty} \int_\Omega \rho(\tau, r)|u_r(\tau, r)|^2 \, dr \right) d\tau. \]

Arguing as in (2.2) we obtain
\[ \|\rho(\tau, \cdot)u^2(\tau, \cdot)\|_{L^\infty} \leq c(1 + \sup_{s \in [0, \tau]} \|\rho(s, \cdot)\|_{L^\infty}) \leq c \left( 1 + \sup_{s \in [0, \tau]} \int_\Omega |\rho_r(s, r)|^2 \, dr \right). \]

(2.15)

Inserting this into the previous inequality we obtain
\[ \int_{Q_t} |\rho u u_r m_{rr}| \, dr \, d\tau \leq \delta \int_{Q_t} |m_r|^2 \, dr \, d\tau + C_\delta \int_0^t \Phi(\tau) \left( 1 + \sup_{s \in [0, \tau]} \int_\Omega |\rho_r(s, r)|^2 \, dr \right) d\tau, \]

(2.16)

where
\[ \Phi(\tau) = \int_\Omega \rho(\tau, r) \gamma - 2 \, |\rho_r(\tau, r)|^2 \, dr + \rho(\tau, r)|u_r(\tau, r)|^2 \, dr \]
is an \( L^1(0, T) \) function with the norm depending on \( a, \varepsilon, E_0 \), see (2.3). Consider now
\[ \left| \int_{Q_t} \frac{2\rho u^2}{r} m_{rr} \, dr \, d\tau \right| \leq \delta \int_{Q_t} |m_r|^2 \, dr \, d\tau + C_\delta \int_0^t \left( \|\rho u^2(\tau, \cdot)\|_{L^\infty} \int_\Omega \rho u^2(\tau, r) \, dr \right) \]
\[ \leq \delta \int_{Q_t} |m_r|^2 \, dr \, d\tau + C_\delta \int_0^t \left( 1 + \sup_{s \in [0, \tau]} \int_\Omega |\rho_r(s, r)|^2 \, dr \right) d\tau, \]
where in the last inequality we used (2.15) and (2.3). All other terms in (2.11) can be estimated similar arguments and we obtain:

\[
\sup_{\tau \in [0,t]} \int_{\Omega} \left| \rho_{r}(\tau, s) \right|^2 + \left| m_{rr}(\tau, s) \right|^2 d\tau + \varepsilon \int_{Q_t} \left| \rho_{rr} \right|^2 + \left| m_{r\tau} \right|^2 drd\tau \\
\leq \delta \int_{Q_t} \left| \rho_{rr} \right|^2 + \left| m_{r\tau} \right|^2 drd\tau \\
+ C_\delta \int_0^t (1 + \Phi(\tau))(1 + \sup_{s \in [0,\tau]} \int_{\Omega} \left| \rho_{r}(s, r) \right|^2 + \left| m_{r}(s, r) \right|^2 dr) d\tau.
\]

The estimate of the lemma follows. \(\Box\)

**Lemma 2.3.** There is a priori bound for \(\|\rho\|_{L^\infty(Q_T)}\) and \(\|u\|_{L^\infty(Q_T)}\) in terms of the parameters of the problem, \(E_0\) and \(\|\rho_0, u_0\|_{L^\infty(\Omega)}\).

**Proof.** The estimate for \(\|\rho\|_{L^\infty(Q_T)}\) follows from (2.2). From this and lemma 2.1 we obtain estimate on \(\|u\|_{L^\infty(Q_T)}\). \(\Box\)

Let \(\tilde{\rho} \in (0, \bar{\rho})\), and define

\[
\phi(\rho) = \begin{cases} 
\frac{1}{\rho} - \frac{1}{\tilde{\rho}} + \frac{\rho - \tilde{\rho}}{\tilde{\rho}^2}, & \rho < \tilde{\rho} \\
0, & \rho > \tilde{\rho}.
\end{cases}
\]

**Lemma 2.4.** There is \(c\) depending on \(\|u\|_{L^\infty(Q_T)}\), \(\|\phi(\rho_0)\|_{L^1(\Omega)}\) and other parameters of the problem such that

\[
\sup_{\tau \in [0,T]} \int_{\Omega} \phi(\rho(t, \cdot)) d\tau + \int_0^T \int_{\Omega} \rho^{-3} |\rho_{r}|^2 drdt \leq c. \tag{2.17}
\]

**Proof.** Indeed, multiplying the first equation in (1.2) by \(\phi'(\rho)\) we get

\[
\phi_t + (\phi u)_r - \varepsilon (\phi)_{rr} + \frac{n\varepsilon}{\rho^2} |\rho_{r}|^2 \chi_{\{\rho < \tilde{\rho}\}} = \left( \frac{2}{\rho} - \frac{2}{\tilde{\rho}} \right) u_r \chi_{\{\rho < \tilde{\rho}\}} + \frac{nu}{r} \left( \frac{1}{\rho^2} - \frac{1}{\tilde{\rho}^2} \right) \chi_{\{\rho < \tilde{\rho}\}}
\]

\[
+ \frac{n\varepsilon}{r} \left( \frac{1}{\rho^2} - \frac{1}{\tilde{\rho}^2} \right) \rho_r \chi_{\{\rho < \tilde{\rho}\}}.
\]
Integrating the above equation in space and time and using the boundary conditions (1.3) we get

\[
\sup_{t \in [0,T]} \int_{\Omega} \phi(\rho) \, d\rho + \varepsilon \int_0^T \int_{\{\rho < \tilde{\rho}\}} \rho^{-3} |\rho_r|^2 \, d\rho \, dt \\
\leq \left| \int_0^T \int_{\{\rho < \tilde{\rho}\}} \left( \frac{2}{\rho} - \frac{2}{\tilde{\rho}} \right) \rho_r \, d\rho \, dt \right| \\
+ \left| \int_0^T \int_{\{\rho < \tilde{\rho}\}} \frac{n \rho u}{r} \left( \frac{1}{\rho^2} - \frac{1}{\tilde{\rho}^2} \right) \, d\rho \, dt \right| \\
+ \left| \int_0^T \int_{\{\rho < \tilde{\rho}\}} \frac{n \varepsilon \rho_r}{r} \left( \frac{1}{\rho^2} - \frac{1}{\tilde{\rho}^2} \right) \, d\rho \, dt \right| \\
= I_1 + I_2 + I_3. \quad (2.18)
\]

We estimate by integrating by parts,

\[
I_1 \leq 2 \int_0^T \int_{\{\rho < \tilde{\rho}\}} \frac{\rho_r u}{\rho^2} \, d\rho \, dt \leq \frac{\varepsilon}{8} \int_0^T \int_{\{\rho < \tilde{\rho}\}} \rho^{-3} |\rho_r|^2 \, d\rho \, dt + C \varepsilon \int_0^T \int_{\{\rho < \tilde{\rho}\}} \frac{|u|^2}{\rho} \, d\rho \, dt. \quad (2.19)
\]

Since, $\rho^{-1} \leq \phi(\rho)$, for small $\rho$’s, $u$ is $L^\infty$ bounded, and $|\{\rho(t, \cdot) \leq \tilde{\rho}\}|$ is bounded independently of $T$, the last term in the above inequality is bounded by

\[
C \left( 1 + \int_0^T \int_{\Omega} \phi(\rho) \, d\rho \, dt \right)
\]

and thus

\[
I_1 \leq 2 \int_0^T \int_{\{\rho < \tilde{\rho}\}} \frac{\rho_r u}{\rho^2} \, d\rho \, dt \leq \frac{\varepsilon}{8} \int_0^T \int_{\{\rho < \tilde{\rho}\}} \rho^{-3} |\rho_r|^2 \, d\rho \, dt \\
+ C \varepsilon \left( 1 + \int_0^T \int_{\Omega} \phi(\rho) \, d\rho \, dt \right). \quad (2.19)
\]

Also, by the similar arguments

\[
I_2 = \left| \int_0^T \int_{\{\rho < \tilde{\rho}\}} \frac{n}{r} \left( \frac{\rho u}{\rho^2} - \frac{u}{\rho} \right) \, d\rho \, dt \right| \leq C \left( 1 + \int_0^T \int_{\Omega} \phi(\rho) \, d\rho \, dt \right) \quad (2.20)
\]

and

\[
I_3 \leq C \int_0^T \int_{\{\rho < \tilde{\rho}\}} \frac{\varepsilon \rho_r}{\rho^2} \, d\rho \, dt \leq \frac{\varepsilon}{8} \int_0^T \int_{\{\rho < \tilde{\rho}\}} \rho^{-3} |\rho_r|^2 \, d\rho \, dt \\
+ C \varepsilon \left( 1 + \int_0^T \int_{\Omega} \phi(\rho) \, d\rho \, dt \right). \quad (2.21)
\]

Combining the last three estimates in (2.18) and using the Gronwall’s lemma we get an a priori estimate of the lemma.
We have the following corollary:
\[
\int_0^T \| \rho^{-1}(t, \cdot) \|_{L^\infty(\Omega)} \, dt \leq C \left( 1 + \left[ \int_{Q_T} \rho^{-3} |\rho_r|^2 \, dr dt \right]^{1/2} \left[ \int_{Q_T} \phi(\rho) \, dr dt \right]^{1/2} \right)
\]
\[
\leq C \left( 1 + \left[ \int_{Q_T} \rho^{-3} |\rho_r|^2 \, dr dt \right]^{1/2} \right). \tag{2.22}
\]

**Lemma 2.5.** There is $C$ that depends on $\| \phi(\rho_0) \|_{L^1(\Omega)}$ and other parameters as listed in the previous lemma, such that
\[
\int_0^T \| \rho^{-1}(t, \cdot)m_r(t, \cdot) \|_{L^\infty(\Omega)} + \| \rho^{-1}(t, \cdot)\rho_r(t, \cdot) \|_{L^\infty(\Omega)} + \| u_r(t, \cdot) \|_{L^\infty(\Omega)} \, dt \leq C. \tag{2.23}
\]

**Proof.** Indeed, by the Sobolev embedding lemma and (2.22) we get
\[
\int_0^T \| \rho^{-1}m_r \|_{L^\infty(\Omega)} \, dt \leq \int_0^T \| m_r \|_{L^\infty(\Omega)} \| \rho^{-1} \|_{L^\infty(\Omega)} \, dt
\]
\[
\leq C(a, \bar{\rho}, T) \int_0^T \left[ \int_{\Omega} |m_r|^2 \, dr \right]^{1/2}
\times \left( 1 + \left[ \int_{\Omega} \rho^{-3} |\rho_r|^2 \, dr \right]^{1/2} \right) \, dt,
\]
which is bounded by lemma 2.2 and (2.17). The estimate for $\frac{\rho_r}{\rho}$ is the same. The estimate for $u_r$ follows from the previous two estimates, lemma 2.3 and $u_r = \frac{m_r}{\rho} - \frac{u}{\rho}$. \hfill \Box

Now we can obtain uniform estimate for $v = \rho^{-1}$. Notice, that $v$ verifies the inequality
\[
v_t + (u - \frac{2\varepsilon}{r})v_r - \varepsilon v_{rr} \leq (u_r + \frac{2u}{r})v.
\]
By the maximum principle,
\[
\max_{[0,T] \times \Omega} \| v \|_{L^\infty(\Omega)} \leq \max_{[0,T] \times \Omega} \left\{ \| v_0 \|_{L^\infty(\Omega)}, \bar{v} \right\} e^{\int_0^T \| u_r(t, \cdot) + 2u(t, \cdot)r^{-1} \|_{L^\infty(\Omega)} \, dt} \leq \max_{[0,T] \times \Omega} \left\{ \| v_0 \|_{L^\infty(\Omega)}, \bar{v} \right\}, \tag{2.24}
\]
by lemma 2.3 and (2.23).

Estimates of lemma 2.3 and (2.24) are the required a priori estimates and the proof of theorem 2.1 is completed. \hfill \Box

### 3. Proof of theorem 1.1

#### 3.1. Apriori estimates independent of $\varepsilon$. We will need the following estimate.
Lemma 3.1. Let $l = 0, \ldots, n$, and $a_1 \in (a, b)$. There are $c = c(\gamma, a_1, E_0)$ and $P = P(b, \tilde{p})$ such that $\forall T > 0$:

$$\sup_{t \in [0, T]} \int_{a_1}^b \rho(t, \cdot) \gamma^l \, dr \leq c(1 + P).$$

(3.1)

In this estimate, $P(b, \tilde{p})$ is a sum of products of positive powers of $b$ and $\tilde{p}$.

Proof. The proof is based on the energy estimate (2.3). Let $e(\rho) = \rho^\gamma - \tilde{p}^\gamma - \gamma \tilde{p}^{\gamma - 1}(\rho - \tilde{p})$. Notice that because $e(\rho)$ is quadratic in $(\rho - \tilde{p})$ for $\rho$ near $\tilde{p}$, it follows that $\tilde{p}^{\gamma - 1}(\rho - \tilde{p}) \leq c(\gamma)\tilde{p}^2\sqrt{e(\rho)}$ for $\rho \in (\tilde{p}/2, 2\tilde{p})$. Also, there are $c(\gamma)$ and $C(\gamma)$ such that if $\rho > C\tilde{p}$ then $\rho^\gamma \leq c(\gamma)(e(\rho) + \tilde{p}^\gamma)$. Using this $C$ we write

$$\int_{a_1}^b \rho^\gamma \, dr = \int_{(a_1, b) \cap \{\rho \in (\tilde{p}/2, \tilde{p})\}} \rho^\gamma \, dr + \int_{(a_1, b) \cap \{\rho \leq \tilde{p}/2\}} \rho^\gamma \, dr + \int_{(a_1, b) \cap \{\rho \geq C\tilde{p}\}} \rho^\gamma \, dr$$

$$= \int_{(a_1, b) \cap \{\rho \in (\tilde{p}/2, \tilde{p})\}} (\rho^\gamma - \tilde{p}^\gamma - \gamma \tilde{p}^{\gamma - 1}(\rho - \tilde{p})) \, dr + \tilde{p}^\gamma \int_{(a_1, b) \cap \{\rho \in (\tilde{p}/2, \tilde{p})\}} \, dr$$

$$+ \gamma \tilde{p}^{\gamma - 1} \int_{(a_1, b) \cap \{\rho \in (\tilde{p}/2, \tilde{p})\}} (\rho - \tilde{p}) \, dr + \int_{(a_1, b) \cap \{\rho \leq \tilde{p}/2\}} \rho^\gamma \, dr$$

$$+ \int_{(a_1, b) \cap \{\rho \geq C\tilde{p}\}} \rho^\gamma \, dr.$$

Using the arguments from the paragraph preceding the equation and the Young’s inequality we get

$$\int_{a_1}^b \rho^\gamma \, dr \leq c \int_{a_1}^b e(\rho) \, dr + P(b, \tilde{p}),$$

where $c = c(\gamma)$ and $P(b, \tilde{p})$ as in the statement of the claim. Since

$$\int_{a_1}^b e(\rho(t, r)) \, dr \leq a_1^{1-n} \sup_{\tau \in [0, t]} \int_a^b E(\rho(\tau, r), m(\tau, r)) \, dr \leq a_1^{2-l} E_0,$$

by the first energy estimate, the claim follows. \(\square\)

Lemma 3.2. Let $K$ be a compact subset of $(0, +\infty)$ and $T > 0$. Then there is $c = c(K, T, E_0)$, independent of all small $\varepsilon$, such that

$$\int_0^T \int_K \rho^{\gamma+1} \, drdt \leq c.$$

(3.2)

Let $\omega(r)$ be a smooth positive compactly supported on $(a, b)$ function. We multiply the momentum equation by $\omega$ and integrate the result in $r$ over $(r, b)$ to get:

$$\partial_t (\rho u \omega) + ((pu^2 + p)\omega)_r + \frac{2\rho u^2 \omega}{r} - \varepsilon \omega'\left(m_r + \frac{2m}{r}\right) = \left((pu^2 + p) - \varepsilon (m_r + \frac{2m}{r})\right) \omega_r,$$

(3.3)
and
\[
\partial_t \int_r^b \rho u \omega \, dr + \int_r^b \frac{2 \rho u^2 \omega}{r} \, dr + \varepsilon \omega (m_r + \frac{2 m}{r}) \\
= \omega (\rho u^2 + p) + f,
\] (3.4)

where
\[
f = \int_r^b \omega_r (\rho u^2 + p - \varepsilon (m_r + \frac{2 m}{r})) \, dr.
\]

Multiplying the previous equation by \( \rho \) and using the continuity equation we get:
\[
\partial_t \left[ \rho \int_r^b \rho u \omega \, dr \right] + \left[ (\rho u)_r + \frac{2 \rho m}{r} - \varepsilon (\rho r + \frac{2 \rho r}{r}) \right] \int_r^b \rho u \omega \, dr \\
+ \rho \int_r^b \frac{2 \rho u^2 \omega}{r} \, dr + \varepsilon \rho \omega (m_r + \frac{2 m}{r}) = (\rho^2 u^2 + \rho \gamma + 1) \omega + \rho f,
\] (3.5)

and
\[
\left[ \rho \int_r^b \rho u \omega \, dr \right]_t + \left[ \rho u \int_r^b \rho u \omega \, dr \right]_r \\
+ \varepsilon \left\{ -(\rho r + \frac{2 \rho r}{r}) \int_r^b \rho u \omega \, dr + \rho \omega (m_r + \frac{2 m}{r}) \right\} = \rho^{\gamma+1} + f_1,
\] (3.6)

where
\[
f_1 = \rho f - \frac{2 \rho u}{r} \int_r^b \rho u \omega \, dr.
\]

We can write
\[
\{...\} = - (\rho_r \int_r^b \rho u \omega \, dr)_r - \rho_r \rho u \omega - \left( \frac{2 \rho}{r} \int_r^b \rho u \omega \, dr \right)_r - \frac{2 \rho^2 u \omega}{r} \\
+ \frac{2 \rho}{r^2} \int_r^b \rho u \omega \, dr + \rho^2 u_r \omega + \rho_r \rho u \omega + \frac{2 \rho^2 u \omega}{r} \\
= - (\rho \int_r^b \rho u \omega \, dr)_r - (\rho^2 u \omega)_r - \left( \frac{2 \rho}{r} \int_r^b \rho u \omega \, dr \right)_r \\
+ \rho^2 u_r \omega + \frac{2 \rho}{r} \int_r^b \rho u \omega \, dr.
\] (3.7)

It follows then that
\[
\left[ \rho \int_r^b \rho u \omega \, dr \right]_t + \left[ \rho u \int_r^b \rho u \omega \, dr \right]_r - \varepsilon (\rho \int_r^b \rho u \omega \, dr)_r - \varepsilon (\rho^2 u \omega)_r \\
- \varepsilon \left( \frac{2 \rho}{r} \int_r^b \rho u \omega \, dr \right)_r + \varepsilon \rho^2 u_r \omega = \rho^{\gamma+1} \omega + f_2,
\] (3.8)
where

\[ f_2 = f_1 - \frac{2\rho}{r} \int_r^b \rho u \omega \, dr. \]

We multiply the last equation by \( \omega \) and write it in the form

\[
\left[ \rho \omega \int_r^b \rho u \omega \, dr \right]_t + \left[ \rho \omega \int_r^b \rho u \omega \, dr \right]_r - \varepsilon \omega \left( \rho \int_r^b \rho u \omega \, dr \right)_r + \left( \varepsilon \rho \omega \int_r^b \rho u \omega \, dr \right)_r
- \varepsilon (\rho^2 u \omega^2)_r - \varepsilon \left( \frac{2\rho \omega}{r} \int_r^b \rho u \omega \, dr \right)_r + \varepsilon \rho^2 u \omega^2 + \varepsilon \rho^2 u \omega \omega_r = \rho^{\gamma + 1} \omega^2 + f_3(3.9)
\]

where

\[ f_3 = \omega f_2 + \rho \omega r \int_r^b \rho u \omega \, dr + \varepsilon \rho \omega r \int_r^b \rho u \omega \, dr - \frac{2\rho \omega r}{r} \int_r^b \rho u \omega \, dr. \]

We integrate the previous equation over \([0, T] \times [a, b]\) to get

\[
\int_0^T \int_a^b \rho^{\gamma + 1} \omega^2 \, dr dt \leq \int_0^T \int_a^b (\varepsilon \rho^2 u \omega^2 + \varepsilon \rho^2 u \omega \omega_r) \, dr dt \\
+ \int_a^b \left( \rho \omega \int_r^b \rho u \omega \, dr \right)|_0^T \, dr - \int_0^T \int_a^b f_3 \, dr dt
\leq \varepsilon \int_0^T \int_a^b \rho^2 \omega^2 \, dr dt + \varepsilon \int_0^T \int_a^b \rho |u_r|^2 \omega^2 + \rho |u|^2 |\omega_r|^2 \, dr dt \\
+ \int_a^b \left( \rho \omega \int_r^b \rho u \omega \, dr \right)|_0^T \, dr - \int_0^T \int_a^b f_3 \, dr dt
\leq \varepsilon \int_0^T \int_a^b \rho^3 \omega^2 \, dr dt + c(supp \omega, T, E_0). \tag{3.10}
\]

The last inequality follows easily from the (2.3), (2.4) and the formula for \( f_3 \).

**Claim 3.1.** Let \( \gamma > 1 \). There is \( C = C(supp \omega, T, E_0) \) such that

\[ \varepsilon \int_0^t \int_a^b \rho^3 \omega^2 \leq C + C \varepsilon \int_0^t \int_a^b \rho^{\gamma + 1} \omega^2. \]
Proof. Let's prove the claim. If $\gamma \geq 2$ the claim is trivial. Let $\gamma < \alpha \leq 3$ and estimate:

$$
\varepsilon \int_0^T \int_a^b \rho^\alpha \omega^2 \leq \varepsilon \sup_{\text{supp}\omega} (\rho^{\alpha-\gamma} \omega^2) \int_0^T \int_{\text{supp}\omega} \rho^\gamma \leq \varepsilon C \sup_{\text{supp}\omega} (\rho^{\alpha-\gamma} \omega^2 ) \\
\leq \varepsilon C \int_0^T \int_a^b \rho^{\alpha-\gamma-\gamma/2}\omega^2 + \varepsilon C \int_0^T \int_a^b \rho^{\alpha-\gamma} \omega \omega_r |
$$

$$
\leq \varepsilon C \left( \int_0^T \int_{\text{supp}\omega} \rho^\gamma + \int_0^T \int_a^b \omega_r^2 \rho^\gamma + \int_0^T \int_a^b \rho^{2\alpha-3\gamma} \omega^2 \right) \\
\leq \varepsilon C \left( 1 + \int_0^T \int_a^b \rho^{2\alpha-3\gamma} \omega^2 \right). \quad (3.11)
$$

If $2\alpha - 3\gamma \leq \gamma + 1$ the estimate of the claim follows. If not, since $2\alpha - 3\gamma < \alpha$ (note that $\alpha \leq 3$), we can iterate the previous estimate with $\alpha$ replaced by $2\alpha - 3\gamma$ and improve the previous estimate:

$$
\varepsilon \int_0^T \int_a^b \rho^\alpha \omega^2 \leq \varepsilon C \left( 1 + \int_0^T \int_a^b \rho^{4\alpha-9\gamma} \omega^2 \right). \quad (3.12)
$$

If $4\alpha - 9\gamma$ is still bigger than $\gamma + 1$ we iterate the estimate again. In this way we get a recurrence relation $\alpha_n = 2\alpha_{n-1} - 3\gamma$, $\alpha_0 = \alpha \leq 3$ and the estimate

$$
\varepsilon \int_0^T \int_a^b \rho^\alpha \omega^2 \leq \varepsilon C(n) \left( 1 + \int_0^T \int_a^b \rho^{\alpha_n-\gamma} \omega^2 \right).
$$

Solving the recurrence relation it we get $\alpha_n = 2^n \alpha - 3\gamma(2^{n-1} - 1)$. For some $n$ the expression is less than $\gamma + 1$ (note that $\alpha \leq 3$) and the estimate of the claim is proved. $\square$

Now returning to (3.10) we obtain:

$$
\int_0^T \int_a^b \rho^{\gamma+1} \omega^2 \leq C(\text{supp}\omega, T, E_0),
$$

for suitably small $\varepsilon$'s.

Let $(\eta, q)$ be a weak entropy/entropy flux pair for one-dimensional Euler system, i.e.,

$$
\eta(\rho, m) = \rho \int_{-1}^1 \psi(m/\rho + \rho^\theta s)[1 - s^2]^{\lambda} ds, \quad (3.13)
$$

$$
q(\rho, m) = \rho \int_{-1}^1 (u + \theta \rho^\theta s)\psi(m/\rho + \rho^\theta s)[1 - s^2]^{\lambda} ds, \quad (3.14)
$$

with $\lambda = (3 - \gamma)/2(\gamma - 1)$ and a density function $\psi(s)$.

The following lemma holds for weak entropies $\eta$, [1].
Lemma 3.3. Let $E(\rho, m)$ be the mechanical energy and $(\eta_\psi, q_\psi)$ be an entropy pair with the generating function

$$\sup_s |\nabla^2 \psi(s)| < \infty.$$  

Then, for any $(\rho, m) \in \mathbb{R}^3_+$ and any vector $\tilde{a} = (a_1, a_2)^T$ it holds:

$$|\langle \nabla^2 \eta(\rho, m)\tilde{a}, \tilde{a} \rangle| \leq C_\psi \langle \nabla^2 E(\rho, m)\tilde{a}, \tilde{a} \rangle,$$  

(3.15)

for some $C_\psi > 0$.

Lemma 3.4. Let $K \subset (0, +\infty)$ be a compact. There is $c = c(K, T, E_0)$, such that for all small $\varepsilon$,:

$$\int_0^T \int_K (\rho|u|^3 + \rho^\gamma + \theta) \, dr dt \leq c(1 + P(b, \bar{\rho})),$$

where $P$ is described in lemma 3.1.

Proof. Let $(\bar{\eta}, \bar{q})$ be a entropy pair corresponding to $\psi = \frac{1}{2} |s|^2$. Define

$$\bar{\eta}(\rho, m) = \bar{\eta}(\rho, m) - \nabla_{\rho, m}\bar{\eta}(\bar{\rho}, 0) \cdot (\rho - \bar{\rho}, m),$$

$$\bar{q}(\rho, m) = \bar{q}(\rho, m) - \nabla_{\rho, m}\bar{\eta}(\bar{\rho}, 0) \cdot (m, m^2/\rho + p).$$

$(\bar{\eta}, \bar{q})$ is an entropy/entropy flux pair. We multiply the continuity equation by $\bar{\eta}_\rho r^2$, the momentum equation by $\bar{\eta}_m r^2$ and add them:

$$(\bar{\eta} r^n)_t + (\bar{q} r^n)_r + nr(-\bar{q} + m\bar{\eta}_\rho + \frac{m^2}{\rho} \bar{\eta}_m + \bar{\eta}_m(\bar{\rho}, 0)p(\rho))$$

$$= \varepsilon r^n \left[ (\rho_{rr} + \frac{n\rho r}{r})\bar{\eta}_\rho + (m_r + \frac{nm}{r}) \bar{\eta}_m \right].$$  

(3.16)

It can be checked directly that

$$\bar{q}(\rho, m) \geq c(\rho|u|^3 + \rho^\gamma + \theta) - c(\rho + \rho|u|^2 + \rho^\gamma),$$

(3.17)

$$-\bar{q} + m(\bar{\eta}_\rho + w\bar{\eta}_m) \leq 0.$$  

(3.18)

Moreover, we have

Claim 3.2. The following inequalities hold pointwisely $\rho, m \in \mathbb{R}^2_+$ with a suitable choice of $c = c(\gamma)$ and $u = m/\rho$.

$$|\bar{\eta}_\rho| \leq c(|u| + \rho^\theta), \quad |\bar{\eta}_m| \leq c(|u|^2 + \rho^{2\theta}),$$

(3.19)

$$|\bar{\eta}| \leq c(\rho + \rho|u|^2 + \rho^\gamma), \quad \rho|\bar{\eta}_\rho| + w\bar{\eta}_m \leq c(\rho + \rho|u|^2 + \rho^\gamma).$$

(3.20)

For $\bar{\eta}_\rho + w\bar{\eta}_m$ considered as a function of $(\rho, u)$, it holds:

$$[\bar{\eta}_\rho + w\bar{\eta}_m]_\rho \leq c(\rho^{\theta-1}|u|^2 + \rho^{2\theta-1}), \quad [\bar{\eta}_\rho + w\bar{\eta}_m]_u \leq c(|u| + \rho^\theta).$$

(3.21)
Proof is direct and is omitted.
Note that at \( r = b, \)
\[
\tilde{q}(\bar{\rho}, 0) = \tilde{\phi}(\bar{\rho}, 0) = c_0(\theta)\bar{\rho}^{\gamma + \theta}, \quad |\tilde{\eta}_m(\bar{\rho}, 0)| = c_1(\theta)\bar{\rho}^\theta, \quad \tilde{\eta}_\rho(\bar{\rho}, 0) = 0,
\]
for some positive \( c_1(\theta). \)

We integrate equation (3.16) over \((0, T) \times (r, b), r \in (a_1, b) \subset (a, b),\) to get:
\[
\int_0^T \tilde{q}(\tau, r) r^n d\tau = c(\theta)\bar{\rho}^{\gamma + \theta} b^n T + \int_r^b (\tilde{\eta}(T, r) - \tilde{\eta}(0, r)) r^n dr \\
+ \int_0^T \int_r^b \left( -\tilde{q} + m\tilde{\eta}_\rho + \frac{m^2}{\rho} \tilde{\eta}_m \right) nr^{n-1} dr dt \\
+ \int_0^T \int_r^b nr^{n-1} \tilde{\eta}_m(\bar{\rho}, 0) p(\rho) dr d\tau \\
+ \int_0^T \int_r^b \varepsilon r^n \left[ (\rho_{rr} + \frac{n\rho_r}{r})\tilde{\eta}_\rho + (m_r + \frac{nm}{r})r\tilde{\eta}_m \right] dr d\tau \\
= I_1 + \ldots + I_5.
\] (3.23)

Lets estimate terms in (3.23). By \( P(b, \bar{\rho}) \) we will denote a generic function with the properites defined in lemma 3.1. Trivially, \( |I_1| \leq c(T)P(b, \bar{\rho}). \) Then, we notice that \( |\tilde{\eta}(\rho, m)| \leq E(\rho, m). \) It follows then that
\[
|I_2| \leq 2 \sup_{\tau \in [0, T]} \int_r^b |\tilde{\eta}(\rho(\tau, r), m(\tau, r))| r^n dr \leq 2 \sup_{\tau \in [0, T]} c \int_a^b E(\rho(\tau, r), m(\tau, r)) r^n dr.
\]

By the energy estimate (2.4), \( |I_2(t, r)| \leq c(E_0). \) Term \( I_3 \) is non positive by (3.18) and can be dropped. Using claim 3.1,
\[
|I_4(t, r)| \leq c(a_1, T)(1 + P(b, \bar{\rho})), \quad \forall (t, r) \in [0, T] \times [a_1, b].
\] (3.24)

It remains to estimate \( I_5. \) We write
\[
r^n(\rho_{rr} + \frac{n\rho_r}{r})\tilde{\eta}_\rho = (r^n\rho_r, \tilde{\eta}_\rho, r^n(m_r + \frac{nm}{r})r\tilde{\eta}_m = (r^n m_r, r\tilde{\eta}_m - nr^{n-2}m\tilde{\eta}_m),
\]
and using the integration by parts (note that \( \tilde{\eta}_\rho(\bar{\rho}, 0) = \tilde{\eta}_m(\bar{\rho}, 0) = 0 \)):
\[
I_5 = -\varepsilon \int_0^t \int_r^b (\rho, \tilde{\eta}_\rho, r m_r\tilde{\eta}_m, r) r^n dr dt - n\varepsilon \int_0^t \int_r^b m\tilde{\eta}_m r^{n-2} dr dt \\
+ \varepsilon \int_0^t \tilde{\eta}_\rho(\tau, r) r^n d\tau = J_1 + J_2 + J_3.
\] (3.25)

\( J_1 \) is estimated by the first energy estimate (2.4) using lemma 3.3: \( |J_1(t, r)| \leq c(\gamma)E_0. \) Also, using claim 3.2 and (3.22) we get
\[
|m\tilde{\eta}_m| \leq c(\rho|u|^2 + \rho_? + \rho|m(\bar{\rho}, 0)|) \leq c(\eta^*(\rho, m) + \bar{\rho}^2 \rho).
\]
It follows by the energy estimate (2.3) that

$$|J_2(t,r)| \leq c(a_1,T,E_0)(1 + P(b,\bar{\rho})).$$

We write

$$\bar{n}_r = (\rho_r\bar{\eta}_\rho + m_r\bar{\eta}_m) = \rho_r(\tilde{\eta}_\rho + u\tilde{\eta}_m) + \rho\tilde{\eta}_m u_r,$$

and for a positive, smooth, compactly supported on \((a_1,b)\) function \(\omega(r)\), we consider

the integral

$$\int_a^b J_3\omega \, dr = \varepsilon \int_0^t \int_a^b \rho(\tilde{\eta}_\rho + u\tilde{\eta}_m)\omega_r \, r^n \, dr \, dt - n\varepsilon \int_0^t \int_a^b \rho(\tilde{\eta}_\rho + u\tilde{\eta}_m)\omega \, r^{n-1} \, dr \, dt$$

$$+ \varepsilon \int_0^t \int_a^b (-\rho_r(\tilde{\eta}_\rho + u\tilde{\eta}_m)_\rho - \rho u_r(\tilde{\eta}_\rho + u\tilde{\eta}_m)_u + \rho\tilde{\eta}_m u_r)\omega \, r^n \, dr \, dt.$$

Noticing that \(\tilde{\eta}_\rho + u\tilde{\eta}_m = \tilde{\eta}_\rho + u\tilde{\eta}_m + \text{const}\), using claim 3.2 and estimates (2.3), (2.4), and claim 3.1 we get from above equation:

$$|\int_a^b J_3(t,r) \omega \, dr| \leq c(a_1,T,E_0,\|\omega\|_{C^1},\delta) + \delta \int_0^t \int_a^b (\rho|u|^3 + \rho^{\gamma+\theta})\omega \, r^n \, dr \, dt.$$

Finally we multiply equation (3.23) by \(\omega\), integrate it over \((a,b)\) and use the estimate (3.17) together with the above estimates for \(I_i\)'s, \(J_i\)'s, and an appropriate choice of \(\delta\) to get

$$\int_0^t \int_a^b (\rho|u|^3 + \rho^{\gamma+\theta})\omega \, r^n \, dr \, dt \leq c(1 + P(b,\bar{\rho}))$$

$$+ \frac{1}{2} \int_0^t \int_a^b (\rho|u|^3 + \rho^{\gamma+\theta})\omega \, r^n \, dr \, dt,$$

from which the statement of the lemma follows. \(\square\)

### 3.2. Weak entropy estimates.

Let \(a = a(\varepsilon) \to 0\), \(b = b(\varepsilon) \to +\infty\), and \(\bar{\rho} = \bar{\rho}(\varepsilon) \to 0\) as \(\varepsilon \to 0\), such that \(P(b(\varepsilon),\bar{\rho}(\varepsilon)) \to 0\), where \(P(b,\bar{\rho})\) is a function appearing in lemma and 3.4. Given a sequence of initial data as in Theorem 1.1, let \((\rho^\varepsilon,m^\varepsilon)\) by the corresponding solution of the viscosity equations (1.2) on \(Q = [0, +\infty) \times [a(\varepsilon),b(\varepsilon)]\) with \(\bar{\rho} = \bar{\rho}(\varepsilon)\), as above. In this section \((\eta,q)\) is an entropy pair from (3.13), (3.14) with \(\psi(s)\) - smooth, compactly supported on \(\mathbb{R}\) function. Let

$$\eta^\varepsilon = \eta(\rho^\varepsilon,m^\varepsilon), \ q^\varepsilon = q(\rho^\varepsilon,m^\varepsilon), \ m^\varepsilon = \rho^\varepsilon u^\varepsilon.$$

We compute

$$\eta^\varepsilon_t + q^\varepsilon_r = -\frac{n}{r} \rho u^\varepsilon [\eta^\varepsilon_\rho + u^\varepsilon \eta^\varepsilon_m] + \varepsilon \left[ \frac{n \rho^\varepsilon}{r} \eta^\varepsilon_\rho + \left( \frac{nm^\varepsilon}{r} \right)_r \eta^\varepsilon_m \right]$$

$$- \varepsilon \rho^\varepsilon_r (\eta^\varepsilon_\rho)_r - \varepsilon m^\varepsilon_r (\eta^\varepsilon_m)_r + \varepsilon \eta^\varepsilon_{rrr}$$

$$= I_1^\varepsilon + \ldots + I_4^\varepsilon. \quad (3.26)$$

We have

$$|I_1^\varepsilon(t,r)| \leq c\rho^\varepsilon|u^\varepsilon|(1 + (\rho^\varepsilon)^\theta) \leq c(\rho^\varepsilon|u^\varepsilon|^2 + \rho^\varepsilon + (\rho^\varepsilon)^\gamma), \quad (3.27)$$
bounded, independently of ε in $L^1(0, T; L^1_{loc}(0, +\infty))$ (all functions are extended by 0 outside (a, b)). Next,

$$I_2^\varepsilon = \varepsilon \left[ \left( \frac{n}{r} \eta^\varepsilon \right)_r + \frac{n}{r^2} (\eta^\varepsilon - m^\varepsilon m^\varepsilon) \right].$$

(3.28)

Since,

$$|\eta^\varepsilon - m^\varepsilon m^\varepsilon| \leq C_\psi (\rho^\varepsilon + \rho^\varepsilon |u^\varepsilon|^2),$$

and

$$\frac{n}{r^2} |\eta^\varepsilon - m^\varepsilon m^\varepsilon|$$

is bounded in $L^1(0, T; L^1_{loc}(0, +\infty))$ independently of ε. On the other hand, if ω is smooth, compactly supported on $\mathbb{R}^2_+$, then

$$\varepsilon \left| \int \int \left( \frac{n}{r} \eta^\varepsilon \right)_r \omega(t, r) drdt \right| = \varepsilon \left| \int \int \frac{n}{r} \eta^\varepsilon \omega_r drdt \right|$$

(3.29)

$$\leq C(\text{supp } \omega) \varepsilon \| \rho^\varepsilon \|_{L^{7+1}(\text{supp } \omega)} \| \omega \|_{W^{1,2}(\mathbb{R}^2_+)}.$$  

(3.30)

Since $\| \rho^\varepsilon \|_{L^{7+1}(\text{supp } \omega)}$ is bounded independently of ε, (see (3.2)), the above estimate shows that

$$\left( \frac{n}{r} \eta^\varepsilon \right)_r \to 0, \quad W^{-1,2}_{loc}(\mathbb{R}^2_+).$$

(3.31)

Thus,

$$I_2^\varepsilon = f^\varepsilon + g^\varepsilon,$$  

(3.32)

where $f^\varepsilon$ is bounded in $L^1(0, T; L^1_{loc}(0, +\infty))$ and $g^\varepsilon \to 0$ in $W^{-1,2}_{loc}(\mathbb{R}^2_+)$. To show that $I_2^\varepsilon \to 0$ in $W^{-1,2}_{loc}$ use the following claim, adopting the arguments of [4].

**Claim 3.3.** Let $K$ is a compact subset of $(0, +\infty)$. Then for any $0 < \delta < 1$, $\varepsilon > 0$,

$$\int_0^T \int_K \varepsilon^{3/2} |\rho^\varepsilon_r|^2 drdt \leq C(\sqrt{\varepsilon} \delta^{\frac{3}{2}} + \delta).$$

(3.33)

In particular $\int_0^T \int_K \varepsilon^{3/2} |\rho^\varepsilon_r|^2 drdt \to 0$, and $\varepsilon \eta^\varepsilon_t \to 0$, in $L^p(0, T; L^p_{loc}(0, +\infty))$, for $p = 2 - 2/(\gamma + 1) \in (1, 2)$.

Assuming this claim, we obtain that $I_2^\varepsilon \to 0$ in $W^{-1,2}_{loc}$ and collecting all the information obtained above we arrive at the main hypothesis of the compensated compactness method:

$$\eta^\varepsilon_t + q^\varepsilon = f^\varepsilon + g^\varepsilon,$$  

(3.34)

where $f^\varepsilon$ is bounded in $L^1(0, T; L^1_{loc}(0, +\infty))$ and $g^\varepsilon \to 0$ in $W^{-1,2}_{loc}(\mathbb{R}^2_+)$, for some $p \in (1, 2)$.

Let now prove the claim 3.3.

**Proof.** For the simplicity of notation we suppress superscript $\varepsilon$ in all functions. Define

$$\phi(\rho) = \begin{cases} \rho^2/2, & \rho < \delta, \\
\delta^2/2 + \delta(\rho - \delta), & \rho \geq \delta, \end{cases}$$

for $\delta > 0$.
so that \( \phi''(\rho) = \chi_{\rho < \delta} \) and \( \rho \phi'(\rho) - \phi(\rho) = \rho^2/2 \), for \( \rho < \delta \) and \( \rho \phi'(\rho) - \phi(\rho) = \delta^2/2 \), for \( \rho \geq \delta \). Let also \( \omega(r) \) be a non-negative smooth compactly supported function on \( (0, +\infty) \). We compute from the continuity equation:

\[
(\phi \omega)_t + (\phi u \omega)_r - \phi u \omega_r - \left( \frac{\rho^2}{2} \chi_{\rho < \delta} + \frac{\delta^2}{2} \chi_{\rho > \delta} \right) \omega u_r + \frac{n\rho u}{r} \min \{\rho, \delta\} \leq \varepsilon (\phi' \omega \rho_r)_r - \varepsilon \min \{\rho, \delta\} \omega' \rho_r + \frac{n\varepsilon}{r} \omega \min \{\rho, \delta\} \rho_r - \varepsilon \omega |\rho_r|^2 \chi_{\rho < \delta}.
\]

Integrating over \( (0, T) \times (0, +\infty) \) we obtain:

\[
\int_0^T \int \varepsilon |\rho_r|^2 \chi_{\rho < \delta} \, dr \, dt = - \int \phi \omega \bigg|_0^T \, dr + \int_0^T \phi u \omega_r \, dr + \int_0^T \bigg( \frac{n\rho u}{r} \min \{\rho, \delta\} \bigg) \, dr \, dt \leq \int_0^T \int \varepsilon \min \{\rho, \delta\} \omega' \rho_r \, dr \, dt + \int_0^T \int \frac{n\varepsilon}{r} \omega \min \{\rho, \delta\} \rho_r \, dr \, dt = I_1 + \ldots + (3.36)
\]

We estimate the integrals on the right.

\[
|I_1| \leq c(\text{supp}\omega)(\delta^2 + \delta) \int_0^T \int_{\text{supp}\omega} \rho \, dr \, dt \leq c(\text{supp}\omega, T, E_0, M_0)\delta.
\]

\[
|I_2| \leq \int_0^T \int_{\text{supp}\omega} \delta |pu| \chi_{\rho < \delta} + (\delta^2 + \delta \rho)|u| \chi_{\rho > \delta} \, dr \, dt \leq \delta \int_0^T \int_{\text{supp}\omega} \rho + \rho |u|^2 \leq c(T, \text{supp}\omega, E_0, M_0)\delta.
\]

\[
|I_3| \leq \frac{\delta^{5/2}}{\sqrt{\varepsilon}} \int_0^T \int_{\text{supp}\omega} \rho + \varepsilon |u_r|^2 \, dr \, dt \leq c(T, \text{supp}\omega, E_0, M_0) \frac{\delta}{\sqrt{\varepsilon}}.
\]

\[
|I_4| \leq c(\text{supp}\omega) \delta \int_0^T \int_{\text{supp}\omega} \rho + \rho |u|^2 \, dr \, dt \leq c(\text{supp}\omega, T, E_0, M_0)\delta.
\]

\[
|I_5| \leq \frac{\sqrt{\varepsilon}}{\delta^{5/2}} \int_0^T \int_{\text{supp}\omega} \rho \varepsilon |\rho_r| \, dr \, dt + \varepsilon \int_0^T \int_{\text{supp}\omega} \rho |\rho_r| \chi_{\rho < \delta} \omega_r \, dr \, dt
\]

\[
\leq \frac{\varepsilon}{4} \int_0^T \int_{\text{int}} |\rho_r|^2 \omega \, dr \, dt + 2\varepsilon \int_0^T \int_{\text{supp}\omega} \rho^2 |\omega_r|^2 \, dr \, dt + \frac{\sqrt{\varepsilon}}{\delta^{5/2}} c(T, \text{supp}\omega, E_0, M_0)
\]

\[
\leq \frac{\varepsilon}{4} \int_0^T \int |\rho_r|^2 \omega \, dr \, dt + \delta c(T, \text{supp}\omega, E_0, M_0) + \frac{\sqrt{\varepsilon}}{\delta^{5/2}} c(T, \text{supp}\omega, E_0, M_0)(3.41)
\]

\( I_6 \) is estimated in the same way as \( I_5 \). Thus the (3.33) is proved. Notice, that

\[
|\eta_r| \leq c(|\rho_r|(|\eta_m| + |u|u_r|) + |\rho u_r|) \leq c(|\rho_r|(1 + \rho^\theta) + |\rho u_r|).
\]
Let \( p \in [1, 2] \), to be chosen later on and compute
\[
\int_0^T \int_K \varepsilon^p |\eta_r|^p \, dr \, dt \leq c \int_0^T \int_K \varepsilon^p |\rho_r|^p \, dr \, dt + \int_0^T \int_K \varepsilon^p |\rho|^{\theta} + \rho |u_r|^p \, dr \, dt \\
\leq \delta + c_5 \int_0^T \int_K \varepsilon^{2p} |\rho_r|^2 \, dr \, dt \\
\quad + c \int_0^T \int_K \varepsilon^p \rho^{p/2} (|\rho|^{7/2} |\rho_r|^p + |\rho^{1/2} u_r|^p) \, dr \, dt \\
\leq \delta + c_5 \int_0^T \int_K \varepsilon^{3/2} |\rho_r|^2 \, dr \, dt \\
\quad + \varepsilon^{p-1} c \int_0^T \int_K \varepsilon (\rho^{\gamma-2} |\rho_r|^2 + \rho |u_r|^2) + \varepsilon \rho^{\gamma-1} \, dr \, dt \\
\leq \delta + c_5 \int_0^T \int_K \varepsilon^{3/2} |\rho_r|^2 \, dr \, dt + \varepsilon^{p-1} c(T, K, E_0), \tag{3.43}
\]
if
\[
2/(2-p) = \gamma + 1 \Leftrightarrow p = 2 - 2/(\gamma + 1).
\]
Combining this with the estimate (3.33) we obtain the conclusion of the claim. \( \square \)

3.3. **Strong convergence.** The apriori estimates and compactness properties collected in previous sections imply by the Div-Curl lemma (see [2]) that
\[
(\rho^\varepsilon, m^\varepsilon) \to (\rho, m) \quad \text{a.e. (} t, r \text{)} \in \mathbb{R}^2_+ \quad \text{and in } L^p_{loc}(\mathbb{R}^2_+) \times L^q_{loc}(\mathbb{R}^2_+),
\]
for \( p \in [1, \gamma + 1) \), and \( q \in [1, 3\gamma + 1/(\gamma + 1)] \) because of the uniform bounds (3.2), (3.4) and
\[
|m|^q = \rho^{q/2} |u|^q \rho^{2q} \leq \rho |u|^{q+1},
\]
for \( q = \frac{3(\gamma+1)}{\gamma+3} \). From the same estimates one also obtains the convergence of the energies: \( (m^\varepsilon)^2/\rho^\varepsilon \to (m)^2/\rho \) and \( (\rho^\varepsilon)^\gamma \to \rho^\gamma \) in \( L^1_{loc}(\mathbb{R}^2_+) \). Since the energy \( \eta^*(\rho, m) \) is a convex function, we obtain by passing to the limit in (2.2):
\[
\int_{t_1}^{t_2} \int_{(0, +\infty)} (m^2/\rho + \rho^\gamma)(t, r) r^n \, dr \, dt \leq (t_1 - t_2) \int_{(0, +\infty)} (m_0^2/\rho_0 + \rho_0^\gamma)(t, r) r^n \, dr \, dt,
\]
from which (1.5) follows.

Finally, the energy estimates (2.3), (2.4), estimate of lemmas 3.2, 3.4 imply equi-integrability of a sequence of \( \eta^\varepsilon_0, \eta^\varepsilon, m^\varepsilon \eta^\varepsilon_0, (m^\varepsilon)^2 \eta^\varepsilon_0, m^\varepsilon, \) and \( q^\varepsilon \), for any \( \psi(s) \) – convex with subquadratic growth at infinity \( (|\psi(s)| \leq s^\alpha, \alpha \in [0, 2), \) for \( s \gg 1 \).

Passing to the limit in the equation (3.26) multiplied by \( r^n \) and integrated against a smooth compactly supported on \( (0, +\infty) \times (0, +\infty) \) function we obtain (1.6).
3.4. Limit in the equations. Let \( \omega(t, r) \) be a smooth, compactly supported on \([0, +\infty) \times [0, b(\varepsilon)) \), function with \( \text{supp}\omega(t, \cdot) \subset (a(\varepsilon), +\infty) \), for all \( t \). By multiplying the first equation in (1.2) we obtain (functions \( \rho^\varepsilon, m^\varepsilon \) are extended by 0 outside of \([a(\varepsilon), b(\varepsilon)] \)):

\[
\int_{\mathbb{R}^2_+} \rho^\varepsilon \omega_t + m^\varepsilon \omega_r + \varepsilon \rho^\varepsilon (\omega_{rr} + \frac{n\omega_r}{r}) (r^n dr) dt + \int_{\mathbb{R}} \rho_0^\varepsilon(r) \omega(0, r) r^n dr = 0. \tag{3.44}
\]

Note that by the energy inequality \( \int_{[0,1]} (\rho^\varepsilon)^2 r^n dr \) is bounded independently of \( \varepsilon \), and so there is no concentration of mass at \( r = 0 \). Passing to the limit in the above equation we deduce:

\[
\int_{\mathbb{R}^2_+} \rho \omega_t + m \omega_r (r^n dr) dt + \int_{\mathbb{R}} \rho_0(r) \omega(0, r) r^n dr = 0,
\]

which can be extended to hold for all smooth, compactly supported on \([0, +\infty) \times [0, +\infty) \), functions \( \omega(t, r) \) with \( \omega_r(t, 0) = 0 \). Consider now the momentum equation. Let \( \omega(t, r) \) be a smooth, compactly supported on \([0, +\infty) \times (a(\varepsilon), b(\varepsilon)) \), function.

\[
\int_{\mathbb{R}^2_+} m^\varepsilon \omega_t + \frac{(m^\varepsilon)^2}{\rho^\varepsilon} \omega_r + p(\rho^\varepsilon) (\omega_r + \frac{n\omega_r}{r}) + \varepsilon m^\varepsilon \omega_{rr} r^n dr dt + \int_{\mathbb{R}} m_0^\varepsilon(r) \omega(0, r) r^n dr = 0.
\]

Passing to the limit we obtain:

\[
\int_{\mathbb{R}^2_+} m \omega_t + \frac{(m)^2}{\rho} \omega_r + p(\rho) (\omega_r + \frac{n\omega_r}{r}) r^n dr dt + \int_{\mathbb{R}} m_0(r) \omega(0, r) r^n dr = 0.
\]

This equation can be extended for all smooth compactly supported on \([0, +\infty) \times [0, +\infty) \) functions \( \omega(t, r) \), with \( \omega(t, 0) = \omega_r(t, r) = 0 \), because \( (m^2/\rho + \rho^\gamma)(t, r)r^n \in L^1_{loc}(0, +\infty) \times [0, +\infty) \).

References


