GIBB’S MINIMIZATION PRINCIPLE FOR APPROXIMATE SOLUTIONS OF
SCALAR CONSERVATION LAWS

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ABSTRACT. In this work we study variational properties of approximate solutions of scalar conservations laws. Solutions of this type are described by a kinetic equation which is similar to the kinetic representation of admissible weak solutions due to Lions-Perthame-Tadmor[12], but also retain small scale non-equilibrium behavior. We show that approximate solutions can be obtained from a BGK-type equation with equilibrium densities satisfying Gibb’s entropy minimization principle.

1. INTRODUCTION

1.1. Motivation. We consider a Cauchy problem for a scalar conservation law

\[ \begin{align*}
\partial_t \rho + \text{div}_x A(\rho) &= 0, \quad (x,t) \in \mathbb{R}^{d+1}, \\
\rho(x,0) &= \rho_0(x), \quad x \in \mathbb{R}^d,
\end{align*} \]

where \( A : \mathbb{R} \to \mathbb{R}^d \) is a Lipschitz continuous function. For initial data \( \rho_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \), the problem is uniquely solvable in the class of admissible (entropy) solutions, as was established in [10]. When an admissible solution \( \rho(x,t) \) is represented by a kinetic density as

\[ \rho(x,t) = \int f(x,t,v) \, dv, \]

with

\[ f(x,t,v) = \begin{cases} 
1_{[0,\rho(x,t)]}, & \rho(x,t) \geq 0 \\
-1_{[\rho(x,t),0]}, & \rho(x,t) < 0
\end{cases}, \]

then \( f \) is a weak solution of a kinetic equation

\[ \partial_t f + A'(v) \cdot \nabla_x f = -\partial_v m, \quad \mathcal{D}'(\mathbb{R}^{2d+1}_+), \]

where \( m \) is non-negative Radon measure on \( \mathbb{R}^{2d+1}_+ \). Conversely, any solution of (3) constrained by condition (2) for some \( \rho(x,t) \) defines an admissible weak solution of conservation law in (1), see [12]. Kinetic methods for obtaining admissible solutions originate in works [5, 9]. References [1, 2, 3, 4, 11, 13, 16] is an short list of some representative results of the kinetic approach to solving systems of quasilinear PDEs.

Given a kinetic density \( f \), with \( \rho = \int f \, dv \), we will denote an equilibrium density in (2) by \( \Pi^eq_f \).

Date: March 14, 2016.
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