AN INTEGRAL REPRESENTATION OF LOWER SEMI-CONTINUOUS FUNCTIONS WITH AN APPLICATION TO A KINETIC MODEL IN GAS DYNAMICS. *

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Abstract. Motivated by the theory of kinetic models in gas dynamics, we obtain an integral representation of lower semi-continuous functions on $\mathbb{R}^d$, $d \geq 1$. We use the representation to study the problem of compactness of a family of the solutions of the discrete time BGK model for the compressible Euler equations.

Key words. The Euler Equations, Conservation Laws, Kinetic Models.

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1. Kinetic representation of the Euler equations. We consider the Euler equations describing the motion of a monoatomic gas on $\mathbb{R}^d \times \mathbb{R}^+$:

$$
\begin{aligned}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla(\rho T) &= 0, \\
\partial_t E + \text{div}((E + \rho T)u) &= 0, \\
E &= \frac{1}{2} \rho |u|^2 + \frac{d}{2} \rho T.
\end{aligned}
$$

An equilibrium density $M = M(\rho, u, T; v)$ for (1) is a function that verifies the moments relations for the state

$$(1.1) \quad \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix} = \int \begin{bmatrix} 1 \\ \frac{v}{|v|^2/2} \end{bmatrix} M(\rho, u, T; v) \, dv,$$

and fluxes

$$(1.2) \quad \begin{bmatrix} \rho u_i \\ \rho u_i u_j + \rho T \delta_{ij} \\ (E + \rho T) u_i \end{bmatrix} = \int \begin{bmatrix} v_i \\ \frac{v_j}{|v|^2/2} \end{bmatrix} M(\rho, u, T; v) \, dv,$$

where $\delta_{ij}$ is the Kroneker symbol.

There are several classes $\mathcal{M}_0$ of the equilibrium densities $M(\rho, u, T; v)$ that verify the moments relations. We will use

$$(1.3) \quad M(\rho, u, T; v) = \frac{c_0}{(d+2)^{d/2} T^{d/2}} \rho \chi_d \left( \frac{|u - v|^2}{T} \right),$$

where $c_0 = |B_1|^{-1}$, $B_1$ is the unit ball in $\mathbb{R}^d$, and $\chi_d(r) = \mathbb{1}_{\{|r|<(d+2)/d\}}(r)$, and define

$$\mathcal{M}_0 = \{ M(\rho, u, T; v) : \rho \geq 0, \ u \in \mathbb{R}^d, \ T > 0 \}.$$

The equilibrium density $M(\rho, u, T; v)$ is obtained as the minimizer of the Gibb’s entropy

$$(1.4) \quad S[f] = \sup_{v} f(v),$$

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constrained by
\[
\begin{bmatrix}
\rho \\
u \\
E
\end{bmatrix} = \int \begin{bmatrix}
1 \\
v \\
|v|^2/2
\end{bmatrix} f(v) \, dv.
\]

The minimizer is unique, see Lemma 1.10.1 of [15] for the exposition of these facts.

Note, also, that
\[
S[M(\rho, u, T; \cdot)] = \frac{c_0}{(d+2)^{d/2}} \frac{\rho}{T^{d/2}},
\]
which is the entropy for the system (1).

In the kinetic approach the system of equations (1) is substituted with a kinetic equation for functions \( f = f(x, t, v) \), and \( Q = Q(x, t, v) \):

(1.5) \quad \partial_t f + v \cdot \nabla_x f = Q,

with the constraints for \( f \) and \( Q \):

(1.6) \quad \forall (x, t) \quad f(x, t, \cdot) \in \mathcal{M}_0, \quad \text{and} \quad \int \begin{bmatrix}
1 \\
v \\
|v|^2/2
\end{bmatrix} Q(x, t, \cdot) \, dv = 0.

A kinetic model is an approximation of the problem (1.5), (1.6). Typically, it renders to ignoring the constraint \( f(x, t, \cdot) \in \mathcal{M}_0 \), and choosing “an interaction” potential \( Q = Q_h[f] \), so that it verifies the second constraint in (1.6), with a relaxation parameter \( h > 0 \). The equation becomes:

(1.7) \quad \partial_t f + v \cdot \nabla_x f = Q_h[f].

The first constraint in (1.6) is obtained in the limit \( h \to 0 \). The Bolzmann and the BGK equations (see below) are the equations of such type.

For the later equation

(1.8) \quad Q_h[f] = \frac{M_f - f}{h},

where \( M_f \in \mathcal{M}_0 \) and has the same \((1, v, |v|^2)\) moments as \( f \). A discrete time analog of (1.8) consists of solving transport equation on the time interval of length \( h \), and then projecting the function \( f \to M_f \in \mathcal{M}_0 \).

The theory of BGK models for the Euler equations was developed in [13, 14], where the existence/stability and uniqueness of weak, global in \( t \) solutions was shown, for the case of the equilibrium densities are Gaussians. The discrete BGK models are well-defined by construction.

The convergence of solutions of the BGK model to solutions of the Euler equations for the isentropic gas was proved in [1], in dimension one, assuming uniform bounds on the density and velocity. For multidimensional isentropic Euler equations the hydrodynamic limit to classical solutions was established in [2]. Kinetic models for scalar conservation laws were extensively studied and there is a comprehensive theory for the limit problem \( h \to 0 \), [5, 8, 12].

Let us discuss model (1.7) in some details. Typically, for BGK models, the Gibb’s entropy provides the estimate

(1.9) \quad \int\int_K \int_{\mathbb{R}^d} |M_{f_h} - f_h|^2 \, dv \, dx \, dt \leq Ch, \quad K = \mathbb{R}^d_x \times (0, T), \quad \forall T > 0,
and in addition, the estimates for moments, see for example [3]:

$$\sup_{h,t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (1 + |v|^2) f_h(x,t,v) \, dv \, dx < +\infty.$$  

The estimate (1.9) implies that $f_h(x,t,\cdot)$ approaches set $\mathcal{M}_0$, however it does not rule out the oscillations in $(x,t)$, and $\lim_h f_h$ might not be an element of $\mathcal{M}_0$. The main obstacle in recovering solutions of the Euler equations is the pointwise in $(x,t)$ compactness of the moments

$$\int \phi(v) f_h(x,t,v) \, dv, \quad \phi \in C_0^\infty(\mathbb{R}^{d+1}_{x,t}).$$  

The classical averaging lemmas [6, 7, 10] were developed to address this issue, however the above estimates are weaker than the conditions required for their application: one would need an estimate of order $h^2$ in (1.9).

The purpose of this work is a study the compactness properties of (1.10) using an integral representation of a non-negative function $f$ as a convex combination of the equilibrium densities from $\mathcal{M}_0$:

$$f(v) = d^{d/2} \int M(\rho,u,T;v) \, d\mu$$  

with the measure $\mu \in \text{Prob}(\mathbb{R}^{d+2}_{\rho,u,T})$, that has no mass on the plane $\{T = 0\}$. The main result, Theorem 2.1, asserts the equivalence between the lower semicontinuous functions and functions in (1.11). The proof of Theorem 2.1 is based on the Vitali’s covering lemma, and is presented in the Appendix. Note, that as $T \to 0$, $M(\rho,u,T;v)$ converges to a delta mass in $v$, supported at $v = u$. Any continuous function can be trivially approximated using the delta masses so that

$$f(v) = \int \delta(v-w) \, d\mu, \quad d\mu = f(w) \, dw$$  

holds, and no new information can be extracted from such representation. Contrary to that, $T > 0$ in (1.11), which shows that function $f$ can be expressed as a sum of characteristic functions of balls. It should be noted however that the measure $\mu$ is not unique.

The representation (1.11) is closely related to a well-known problem in Probability of decomposing a probability density function into a simplex of normal densities, see for example the discussion in [11].

The applications to the kinetic models of the Euler equations starts with Lemma 3.1, in which we show that any solution to the transport equation

$$\partial_t f + v \cdot \nabla_x f = 0$$

defines a parametrized measure $\mu_{x,t}$, that verifies certain differential equations in $(x,t)$. In particular, it represents a measure-valued solution of the Euler equations. This result should be viewed as an analog of the multi-valued solutions of scalar conservation laws: when the initial data profile $u(\cdot,0)$ is transported with different speeds, at different locations $x$, the graph of $u(\cdot,t)$ becomes multi-valued.

Next, we consider discrete time BGK model for the Euler equations. Based on the integral representation (1.11) we introduce a notion of a linear kinetic entropy for a family $\{f_h\}$ of solutions of the BGK model.
In Lemma 5.2, we show that the existence of a single linear kinetic entropy is sufficient for the strong compactness of moments (1.10). Here we employ the compactness result of Gérard[7].

Finally, in Lemma 5.4 we state the sufficient conditions, for a family \( \{f_h\} \) to admit a linear, kinetic entropy. We show that the entropy equals \( \delta(v - v^*) \) – delta mass at point \( v^* \), and coincides with the Gibb’s entropy \( S \):

\[
\langle f_h(x,t,v), \delta(v - v^*) \rangle = S[f_h(x,t,v)], \quad \forall (x,t).
\]

2. Main result. Let \( \mathbb{P} = \{(u,T) : u \in \mathbb{R}^d, T > 0\} \), \( \mathcal{M}_+(\mathbb{R}^{d+1}) \) be the set of Radon measures on \( \mathbb{R}^{d+1} \), \( \text{Prob}(\mathbb{R}_{u,T}^{d+1}) \) be the set of unit mass, Radon measures, and \( \mathcal{M}_{+\text{loc}}(\mathbb{R}^{d+1}) \) set of measures that belong to \( \mathcal{M}_+(B_R) \), for any \( R > 0 \), with the topology generated by total variations of a measure on each ball \( B_R \).

**Theorem 2.1.** The following statements are equivalent.

1. \( f(v) \) be a non-negative, unit mass, l.s.c. function on \( \mathbb{R}^d \);
2. there is \( \mu \in \text{Prob}(\mathbb{R}_{u,T}^{d+1}) \) with \( \text{supp} \mu \subset \bar{\mathbb{P}} \), \( \mu(\mathbb{P}) = 1 \), and such that

\[
(2.1) \quad f(v) = c_0 \int \chi \left( \frac{|v - u|^2}{T} \right) \frac{1}{T^{d/2}} d\mu, \quad \text{a.e. } v \in \mathbb{R}^d,
\]

where

\[
\chi(r) = 1_{\{|r| < 1\}}(r),
\]

and \( c_0 = (\text{volume } (B_1))^{-1}, B_1 \) – unit ball.

Note, the kernel in (2.1) has a singularity at \( T = 0 \), but the measure has no mass on that set: \( \mu(\bar{\mathbb{P}} \setminus \mathbb{P}) = 0 \). The integral in (2.1) is understood as

\[
\lim_k c_0 \int_{C_k} \chi \left( \frac{|v - u|^2}{T} \right) \frac{1}{T^{d/2}} d\mu,
\]

where \( C_k \) is a sequence of increasing compact sets \( C_k \subset \mathbb{P} \) such that \( \mathbb{P} = \bigcup_k C_k \). The value of the integral is independent of a choice of such sequence (see Appendix).

3. A representation of solutions of the transport equation. Consider the Cauchy problem

\[
(3.1) \quad \begin{cases}
\partial_t f + v \cdot \nabla_x f = 0 \\
f(x,0,v) = f_0(x,v),
\end{cases}
\]

with the initial data such that \( f_0 \) is non-negative, l.s.c. in \( (x,v) \), and for some \( p \geq 1, p \in \mathbb{N} \),

\[
\iint (1 + |v|^p) f_0(x,v) \, dx dv < +\infty.
\]

The solution of (3.1) equals

\[
f(x,t,v) = f_0(x - tv,v),
\]

it is non-negative and for any \( t > 0 \),

\[
\iint (1 + |v|^p) f(x,t,v) \, dx dv = \iint (1 + |v|^p) f_0(x,v) \, dx dv < +\infty.
\]
Lemma 3.1. For all $(x, t)$ there a measure $\mu_{x,t} \in \text{Prob}(\mathbb{R}^{d+2}_{\rho,u,T})$ such that

\begin{equation}
(3.2) \quad f(x, t, v) = d^{d/2} \int M(\rho, u, T; v) \, d\mu_{x,t}, \quad \text{a.e. } v.
\end{equation}

For any $\phi \in C^\infty(\mathbb{R}^d)$ with $|\phi(v)| \leq C(1 + |v|^{p-1})$, $\mu_{x,t}$ verifies the differential equations:

\begin{equation}
(3.3) \quad \partial_t \langle \eta(\rho, u, T), \mu_{x,t} \rangle + \text{div}_x \langle q(\rho, u, T), \mu_{x,t} \rangle = 0, \quad \mathcal{D}'(\mathbb{R}^{d+1})_x,
\end{equation}

where

\[
\eta = \int \phi(v) M(\rho, u, T; v) \, dv,
\]

\[
q = \int v\phi(v) M(\rho, u, T; v) \, dv.
\]

Proof. It follows from the formula

\[
f(x, t, v) = f_0(x - tv, v),
\]

that for any $t$, $f(x, t, v)$ is l.s.c. in $(x, v)$, and for all $(x, t)$ is l.s.c. in $v$. If $f(x, t) = \int f(x, t, v) \, dv = 0$, define

\[
\mu_{x,t} = \delta(\rho) \otimes \delta(m) \otimes \delta(T - 1).
\]

If $\rho(x, t) > 0$ the function $f/\rho$ is l.s.c. in $v$ and has unit mass. It can be represented by an integral (2.1) with measure $\mu(x, t) \in \text{Prob}(\mathbb{R}^{d+1}_{u,T})$. The kernel in the integral in (2.1) and the equilibrium density (1.3) (with $\rho = 1$) differ only by a scaling of the temperature $T$. Thus, there is a probability measure on $\mathbb{R}^{d+1} (\text{still denoted by } \mu(x, t))$, such that

\[
f(x, t, v)/\rho(x, t) = d^{d/2} \int M(1, u, T; v) \, d\mu(x, t), \quad \text{a.e. } v.
\]

Define measure $\mu_{x,t} \in \text{Prob}(\mathbb{R}^{d+2}_{\rho,u,T})$, by

\[
\mu_{x,t} = \delta(\rho - \rho(x, t)) \otimes \mu(x, t).
\]

Then, (3.2) holds.

Since for any $\phi$ as in the conditions of the lemma,

\[
\int (1 + |v|^p) f \, dv \in L^1_{\text{loc}}(\mathbb{R}^{d+1}_{x,t}),
\]

and $f$ is the solution of the transport equation, it verifies equations

\[
\partial_t \int \phi(v) f \, dv + \text{div}_x \int v\phi(v) f \, dv = 0, \quad \mathcal{D}'(\mathbb{R}^{d+1})_x,
\]

with any test function $\phi$ as in the conditions of the lemma. The equation (3.3) follows from this and the integral representation of $f$ in terms $\mu_{x,t}$. \[\square\]

The lemma shows in particular, that when $p \geq 3$, $\mu_{x,t}$ is a measure-valued solution of the Euler equations. It, certainly, can not be accepted as “physically” meaningful solution, but it can be used to build a more realistic solution when transport is combined with “interaction” in a BGK model.
4. Continuous and discrete time BGK models. The continuous time BGK model for the Euler equations is the Cauchy problem

\[
\begin{cases}
\partial_t f + v \cdot \nabla_x f = \frac{M_f - f}{h} \\
f(x,0,v) = f_0(x,v)
\end{cases}
\]

where \( h > 0 \) is the relaxation parameter and \( M_f \) is the equilibrium density corresponding to \( f(x,t,\cdot) \). The discrete time BGK model is defined by the iterative algorithm. Set

\[
f_h(x,0,v) = f_0(x,v).
\]

For all \( t \in [nh,(n+1)h) \), set

\[
f_h(x,t,v) = f_h(x-tv,nh,v),
\]

and for \( t = (n+1)h \), set

\[
f_h(x,t,v) = M_{f_h(x-v,nh,v)}.
\]

The function \( f_h(x,t,v) \) is well-defined for all \((x,t,v)\), and is periodic in \( x \). It also holds that

\[
\sup_t \int_{\mathbb{R}_d^d} \int_{\Pi} (1 + |v|^2) f_h \, dv \, dx = \text{bounded, independently of } h.
\]

Moreover, \( f_h \) is a solution of the equation

\[
\partial_t f + v \cdot \nabla_x f = \sum_{n=1}^{\infty} \delta(t-nh)(M_f(x-v,(n-1)h,v) - f(x,v,(n-1)h,v)), \quad \mathcal{D}'(\mathbb{R}_{x,t}^{d+1}).
\]

In the rest of the paper we will assume that the initial data \( f_0(x,v) \) and \( f_h(x,t,v) \) are non-negative, periodic in \( x \), with the period \( \Pi = [0,l]^d \), for some \( l > 0 \), and

\[
\int_{\mathbb{R}_d^d} \int_{\Pi} (1 + |v|^2) f_0 \, dv \, dx < +\infty.
\]

5. Linear kinetic entropy. We will describe densities \( f_h \) using representation (2.1), assuming that \( f_h \) is a l.s.c. function for all \((h,x,t)\).

Denote

\[
\rho_h(x,t) = \int f_h \, dv.
\]

If \( \rho_h > 0 \), let \( \mu(h,x,t) \) be a family of measures from theorem 2.1, for function \( f(\cdot) = f_h(x,t,\cdot)/\rho_h(x,t) \). We introduce a probability measure \( \tilde{\mu}_{h,x,t} \in \text{Prob}(\mathbb{R}_{x,t}^{d+2}) \) as

\[
\tilde{\mu}_{h,x,t} = \delta(\rho - \rho_h(x,t)) \otimes \mu(h,x,t).
\]

In this way

\[
f_h(x,t,v) = c_0 \int \frac{\rho}{T^{d/2}} \chi \left( \frac{|v - u|^2}{T} \right) \, d\tilde{\mu}_{h,x,t}.
\]
for all \((h,x,t)\) and a.e. \(v\).

It will be convenient to introduce a version of representation (5.1) using conservative variables \((\rho, m, E)\), where

\[
(5.2) \quad m = \rho u, \quad E = \rho T + \rho |u|^2 / 2.
\]

For \(\rho > 0\) the change of variables \((\rho, u, T) \rightarrow (\rho, m, E)\) is one-to-one, and if we denote by \(\tilde{\mu}_{h,x,t} \in \text{Prob}(\mathbb{R}_{\rho, m, E}^{d+2})\) the image of measure \(\hat{\mu}_{h,x,t}\) from (5.1) under this map we obtain

\[
(5.3) \quad f(x, t, v) = c_0 \int K(\rho, m, E; v) d\hat{\mu}_{h,x,t}, \quad \text{a.e. } v,
\]

where \(K(\rho, m, E; v) = \rho T d/2 \chi \left( \frac{|v-u|^2}{\rho^2} \right) \) when relations (5.2) hold.

**Definition 5.1.** We say that family \(\{f_h\}\) admits a linear kinetic entropy if

1. for all \((h,x,t)\), \(f_h(x,t,\cdot)\) is a l.s.c. function of \(v\);
2. for all \(h\), \(f_h\) has representation (5.3) with measure \(\hat{\mu}_{h,x,t}\), and
   
   \[ \text{supp } \hat{\mu}_{h,x,t} \subset V, \quad \forall (h,x,t), \]
   
   for some compact, convex set \(V \subset \hat{P}\), \(\hat{P} = \{(\rho, m, E) : \rho > 0, E - |m|^2/\rho^2 > 0, m \in \mathbb{R}^d\}\);
3. there is \(\omega \in M_{+}(\mathbb{R}_{v}^d)\), such that

   \[ \Phi(\rho, m, E) = \int K(\rho, m, E; v) d\omega \]

   is twice continuously differentiable, strictly convex function on an open neighborhood of \(V\).

**Theorem 5.2.** Let \(\{f_h\}\) be a family of solutions of (4.1)or(4.2). If the family admits a linear kinetic entropy, then moments

\[
\left\{ \int \phi(v) f_h \, dv \right\} \quad \text{pre-compact in } L^2_{loc}(\mathbb{R}^{d+1}_{x,t}),
\]

for all \(\phi \in C_{0}^{\infty}(\mathbb{R}_{v}^d)\).

**Proof.** We will carry out the proof assuming that \(\omega\) is absolutely continuous measure with respect to Lebesgue measure on \(\mathbb{R}_{v}^d\):

\[ \omega = \omega(v) \, dv, \quad \omega(\cdot) \in L^1(\mathbb{R}_{v}^d). \]

The general case follows by suitable approximation of \(\omega\) by absolutely continuous measures.

We will verify that under the conditions of the theorem a compactness result of Gérard\cite{7} applies. We follow \cite{4} for its presentation.

**Theorem 5.3.** Let \(\Omega\) be an open set of \(\mathbb{R}_{x,t}^{d+1}\), and \(\{f_n\} - \text{bounded sequence in } L^2_{loc}(\Omega \times \mathbb{R}_{v}^d)\) such that

\[
\forall \phi \in C_{0}^{\infty}(\mathbb{R}_{v}^d), \quad \left\{ \int (\partial_t f_n + \text{div}_x(v f_n)) \phi(v) \, dv \right\} \quad \text{pre-compact in } H^{-1}_{loc}(\Omega).
\]
Then, for all $\phi \in C^\infty_0(\mathbb{R}^d)$,

$$\left\{ \int f_n \phi \, dv \right\} \text{ pre-compact in } L^2_{loc}(\Omega).$$

Consider a family $\{f_h\}$ of solutions of (4.2). Choose $\Omega = (0, l)^d \times (0, T)$. From the representation (5.3) and the conditions on the supp $\hat{\mu}_{h,x,t}$, follows that $f_h$ uniformly bounded in $(h, x, t, v)$, and thus $\{f_h\}$ is bounded in $L^2_{loc}(\Omega \times \mathbb{R}^d)$.

Let $\hat{\mu}_{h,x,t}$ be the measures from the representation (5.3) of function $f_h$. From the definition of $f_h$, it follows that

$$\hat{\mu}_{h,x,nh} = \delta(\rho - \rho_h(x, nh-)) \otimes \delta(m - m_h(x, nh-)) \otimes \delta(E - E_h(x, nh-)),$$

where $(\rho_h(x, nh-), m_h(x, nh-), E_h(x, nh-))$ equals to $(1, v, |v|^2)$ moments of $f_h(x, v, nh-) = f_h(x - h v, (n - 1)h, v)$, and also is the center of mass of the measure $\hat{\mu}_{h,x,nh-}$. Denote by

$$\Phi(\rho, m, E) = \int \omega(v) K(\rho, m, E; v) \, dv,$$

the function from the definition of the linear kinetic entropy. Since it is strictly convex, we obtain

$$\int \omega(v) \left( M_{f_h(x - hv, (n - 1)h, v)} - f_h(x - hv, (n - 1)h, v) \right) \, dv$$

$$= -C(\nabla^2 \Phi, V) \int |(\rho - \rho_h(x, nh-), m - m_h(x, nh-), E - E_h(x, nh-))|^2 \, d\mu_{h,x,nh-}.$$

Thus, from the kinetic equation (4.2), we obtain

$$\sum_n \int_{\Pi} |(\rho - \rho_h(x, nh-), m - m_h(x, nh-), E - E_h(x, nh-))|^2 \, d\mu_{h,x,nh-} \, dx$$

$$\leq 2 \sup_t \int_{\Pi} \int |\omega(v)| f_h(x, t, v) \, dv \, dx \leq C,$$

with $C$, independent of $h$. Thus, for any $\phi \in C^\infty_0(\mathbb{R}^d)$, there is $C'$ independent of $h$, such that

$$\int_{\Pi} \int_0^\infty \phi(v) (\partial_t f_h + \text{div}_x(v f_h)) \, dv \, dt \, dx \leq C'.$$

In this way we established that

$$\left\{ \int \phi(v) (\partial_t f_h + \text{div}_x(v f_h)) \, dv \right\} \text{ bounded in } M_{+,loc}(\mathbb{R}^d_x \times \mathbb{R}^d_t).$$
and, since $f_h \in L^\infty_{x,t,v}$,

\begin{equation}
\left\{ \int \phi(v) \left( \partial_t f_h + \text{div}_x(vf_h) \right) dv \right\} \text{ bounded in } W^{-1,\infty}_{loc}(\mathbb{R}^d_x \times \mathbb{R}^+_t).
\end{equation}

By the compactness of the embedding $W^{-1,\infty} \cap \mathcal{M}_{+,loc} \subset H^{-1}_{loc}$,

\begin{equation}
\left\{ \int \phi(v) \left( \partial_t f_h + \text{div}_x(vf_h) \right) dv \right\} \text{ pre-compact in } H^{-1}_{loc}(\mathbb{R}^d_x \times \mathbb{R}^+_t).
\end{equation}

Thus, $\{f_h\}$ verifies the conditions of the Gérard’s theorem and so

\begin{equation*}
\left\{ \int \phi(v)f_h dv \right\} \text{ pre-compact in } L^2_{loc}(\mathbb{R}^d_x \times \mathbb{R}^+_t).
\end{equation*}

\[ \square \]

\[ \square \]

In the following lemma we state conditions on $f_h$, sufficient for the existence of the linear kinetic entropy. It should be noted that the value of that entropy coincides with the value of the Gibb’s entropy $S$, (1.4). In other words, under the conditions of the lemma, the non-linear entropy $S$ has a linear representation. Let $v^* \in \mathbb{R}^d$. Denote an open wedge

\[ W(v^*) = \left\{ (\rho, u, T) \mid \rho > 0, T > 0, |u - v| < T \right\}. \]

Set $W(v^*)$ is the support of the kernel $K(\rho, m, E; v^*)$ from the representation (5.3). For all $(\rho, m, E) \in W(v^*)$, $K(\rho, m, E; v^*) = \rho/T^{d/2}$.

**Lemma 5.4.** Suppose that for some $v^*$, there is a compact, convex set $V \subset W(v^*)$, such that for all $(h, x, t)$:

\[ \text{supp } \hat{\mu}_{h,x,t} \subset V. \]

Then, the family $\{f_h\}$, admits a linear kinetic entropy with $\omega(v) = \delta(v - v^*)$.

**Proof.** Under assumptions of the lemma, for any $(h, x, t)$, $f_h(x, t, v)$ is continuous at $v = v^*$. Thus, $\int \omega f_h dv$ is well-defined and equals to $f_h(x, t, v^*)$. Moreover, the condition on $\text{supp } \hat{\mu}_{h,x,t}$ implies that

\[ f_h(x, t, v^*) = \int \frac{\rho}{T^{d/2}} d\hat{\mu}_{h,x,t}. \]

Moreover, since $d\hat{\mu}_{h,x,t}$ is supported on the plane $\{ \rho = \rho_h(x, t) \}$, we can write

\[ f_h(x, t, v^*) = \frac{1}{\rho_h(x, t)} \int \frac{\rho^2}{T^{d/2}} d\hat{\mu}_{h,x,t}. \]

Since function $\rho^2/T^{d/2}$ is strictly convex function of the conservative variables $(\rho, m, E)$, for $\rho > 0$, see for example [9], the lemma follows. Notice also, that under the condition of the lemma,

\[ f_h(x, t, v^*) = \sup_v f_h(x, t, v) = S[f_h(x, t, \cdot)]. \]

\[ \square \]
6. Appendix. We will follow conventional notation from the set theory and functional analysis: \( \mathbb{1}_A \) – characteristic function of set \( A \); \( \overline{A} \) – closure of set \( A \); \( A^c \) – its complement; \( |A| \) – Lebesgue measure of \( A \).

Proof of Theorem 2. Let us show first that part 2. implies part 1.
For any \( n \in \mathbb{N} \), and \( C_n \) as in the note after the statement of the theorem, consider the integral

\[
c_0 \int_{C_n} \chi \left( \frac{|v-u|^2}{T} \right) \frac{1}{T^{d/2}} dv.
\]

It is a l.s.c. function by the Fatou’s convergence theorem and the fact that \( \chi \left( \frac{|v-u|^2}{T} \right) \frac{1}{T^{d/2}} \) is a l.s.c. in \( v \):

\[
\liminf_{v \to v_0} f(v) = \liminf_{v \to v_0} c_0 \int_{C_n} \chi \left( \frac{|v-u|^2}{T} \right) \frac{1}{T^{d/2}} dv \\
\geq c_0 \int_{C_n} \liminf_{v \to v_0} \chi \left( \frac{|v-u|^2}{T} \right) \frac{1}{T^{d/2}} d\mu \\
\geq c_0 \int_{C_n} \chi \left( \frac{|v_0-u|^2}{T} \right) \frac{1}{T^{d/2}} d\mu.
\]

Since,

\[
f(v) = \lim_{n \to \infty} c_0 \int_{C_n} \chi \left( \frac{|v-u|^2}{T} \right) \frac{1}{T^{d/2}} d\mu = \sup_n c_0 \int_{C_n} \chi \left( \frac{|v-u|^2}{T} \right) \frac{1}{T^{d/2}} d\mu,
\]

\( f(v) \) is a l.s.c. as a supremum of l.s.c. functions. Moreover, \( f(v) \) is clearly non-negative, and since

\[
c_0 \int \chi \left( \frac{|v-u|^2}{T} \right) \frac{1}{T^{d/2}} dv = 1,
\]

for all \((u,T)\), \( f(v) \) also has unit mass.

Let us show that part 1. implies part 2.

Will be using repeatedly the following variant of the Vitali’s covering lemma.

Lemma 6.1. Let \( U \) be an open subset of \( \mathbb{R}^d \). There is a countable collection of open, disjoint balls \( B_i \subset U \), such that

\[
|U \setminus \bigcup_{i=1}^{\infty} B_i| = 0.
\]

In particular, for any \( \varepsilon > 0 \) there is \( N \) such that

\[
|U \setminus \bigcup_{i=1}^{N} B_i| < \varepsilon.
\]

Consider first the case when \( f(v) = |U|^{-1} \mathbb{1}_U(v) \), for an open set \( U \subset \mathbb{R}^d \). Let \( \{B_i\} \) be the collection of balls from the Vitali’s covering lemma. Let \( u_i \) be the center of \( B_i \) and \( T_i \) – square of the radius of \( B_i \). Part 2 of the theorem follows with

\[
\mu = \frac{1}{c_0|U|} \sum_{i=1}^{\infty} T_i^{d/2} \delta(u - u_i, T - T_i),
\]

where \( \delta(u - u_i, T - T_i) \) is the delta-function in \( \mathbb{R}^{d+1} \) at point \((u_i, T_i)\).
Consider a general case. Function $f(v)$ can be represented as

$$f(v) = \int_0^\infty \mathbb{1}_{\{f(\cdot) > \lambda\}}(v) \, d\lambda.$$  

For any $n \in \mathbb{N}$, let $\{\lambda_{n,i}\}_{i=1}^{2^n}$ be the dyadic partition of interval $[0, 2^n]$: 

$$\lambda_{n,i} = \frac{i}{2^n}, \quad i = 1..2^n.$$  

Denote the open set 

$$U_{n,i} = \{f(\cdot) > \lambda_{n,i}\}.$$  

It holds:

$$\int_0^\infty \mathbb{1}_{\{f(\cdot) > \lambda\}}(v) \, d\lambda = \lim_{n \to \infty} \sum_{i=1}^{2^n} \frac{1}{2^n} \mathbb{1}_{U_{n,i}}(v), \quad \text{a.e. } v,$$

since

$$\left| \int_0^\infty \chi(v) \mathbb{1}_{\{f(\cdot) > \lambda\}}(v) \, d\lambda - \sum_{i=1}^{2^n} \frac{1}{2^n} \mathbb{1}_{U_{n,i}}(v) \right| \leq \frac{1}{2^n}, \quad v \in \{f(\cdot) \leq 2^n\}.$$  

We will construct an increasing open cover of sets $U_{n,i}$ by induction. For $n = 1$, $i = 1..4$, and any $\varepsilon_1 > 0$, there are disjoint open balls $B_{1,i}^j \subset U_{1,i}$, $j = 1..N_{1,i}(\varepsilon_1)$, such that

$$|U_{1,i} \setminus \bigcup_j B_{1,i}^j| < \varepsilon_1.$$  

Thus, there is a set $E_1$, $|E_1| < 4\varepsilon_1$, such that

$$\sum_{i=1}^4 \frac{1}{2} \mathbb{1}_{U_{1,i}}(v) - \sum_{i=1}^{N_{1,i}} \frac{1}{2} \mathbb{1}_{B_{1,i}^j}(v) = 0, \quad v \in (E_1)^c.$$  

Suppose that for $i = 1..2^{2(n-1)}$ and $\varepsilon_{n-1}$ there are disjoint open balls $B_{n-1,i}^j \subset U_{n-1,i}$, $j = 1..N_{n-1,i}$,

$$|U_{n-1,i} \setminus \bigcup_j B_{n-1,i}^j| < \varepsilon_{n-1},$$  

and there is a set $E_{n-1}$ of measure $|E_{n-1}| < 2^{2(n-1)}\varepsilon_{n-1}$, such that

$$\sum_{i=1}^{2^{2(n-1)}} \frac{1}{2^{n-1}} \mathbb{1}_{U_{n-1,i}}(v) - \sum_{i=1}^{N_{n-1,i}} \frac{1}{2^{n-1}} \mathbb{1}_{B_{n-1,i}^j}(v) = 0, \quad v \in (E_{n-1})^c.$$  

At the next step, $n$, we will use all balls $B_{n-1,i}^j$, from the step $n-1$, and will add more balls, if necessary. Let $i = 1..2^2n$, and fix $\varepsilon_n > 0$. For sets $U_{n,i}$ with $2^{2(n-1)} < i \leq 2^n$ we use Vitali’s covering lemma to find disjoint open balls $B_{n,i}^j$, contained in $U_{n,i}$ and such that

$$|U_{n,i} \setminus \bigcup_j B_{n,i}^j| < \varepsilon_n.$$
For $1 \leq i \leq 2^{2(n-1)}$ and $i = 2k$, $U_{n,2k} = U_{n-1,k}$, Using balls $B^j_{n,2k} = B^j_{n-1,k}$, $j = 1..N_{n-1,k}$ and adding more balls if necessary we obtain a collection of open disjoint balls $B^j_{n,2k}, j = 1..N_{n,2k}$, such that

$$(6.2) \quad |U_{n,i} \cup \bigcup_{j=1}^{N_{n,i}} B^j_{n,i}| < \varepsilon_n.$$

For $i = 2k - 1$, $U_{n-1,k} \subset U_{n,2k-1}$. Using balls $B^j_{n,2k-1} = B^j_{n-1,k}$, $j = 1..N_{n-1,k}$ and adding more balls if necessary we obtain a collection of open disjoint balls $B^j_{n,2k-1}$, $j = 1..N_{n,2k-1}$, such that inequality (6.2) holds. It follows that there is a set $E_n$ of measure $|E_n| < 2^{2n} \varepsilon_n$, such that

$$(6.3) \quad \sum_{i=1}^{2^{2n}} \frac{1}{2^n} |U_{n,i}| - \sum_{i=1}^{2^{2n}} \sum_{j=1}^{N_{n,i}} \frac{1}{2^n} |B^j_{n,i}| = 0, \quad v \in (E_n)^c.$$

Assuming that $\varepsilon_n$ is so small that $\lim_{n} 2^{2n} \varepsilon_n = 0$, we obtain that (6.3) holds outside of a set of vanishing measure $E_n$, and thus, a.e. $v$,

$$\lim_{n} \sum_{i=1}^{2^{2n}} \sum_{j=1}^{N_{n,i}} \frac{1}{2^n} |B^j_{n,i}| = \int_{0}^{\infty} \mathbb{1}_{\{f(\cdot) > \lambda\}}(v) d\lambda.$$

We use delta-masses to represent

$$\sum_{i=1}^{2^{2n}} \sum_{j=1}^{N_{n,i}} \frac{1}{2^n} |B^j_{n,i}| = \int \chi \left( \frac{|v - u^2_n|}{T} \right) \frac{1}{T^{d/2}} d\mu_n,$$

where

$$\mu_n = \sum_{i=1}^{2^{2n}} \sum_{j=1}^{N_{n,i}} \frac{(T^{d/2}_{n,i})}{2^n} \delta(u - u^j_{n,i}, T - T^{d/2}_{n,i}),$$

$u^j_{n,i}$ being the center of the ball $B^j_{n,i}$, and $T^{d/2}_{n,i}$ - square of its radius. By construction,

$$(6.4) \quad \mu_n = \mu_1 + \sum_{i=2}^{n} \nu_i, \quad n > 1,$$

for some $\nu_i \in \mathcal{M}_+(\mathbb{P})$. Moreover, for all $n > 0$, $\mu_n$ has compact support in $\mathbb{P}$.

From the identities

$$1 = \int f(v) dv = \int_{0}^{\infty} |\{f(\cdot) > \lambda\}| d\lambda = \lim_{n} \sum_{i=1}^{2^{2n}} \frac{1}{2^n} |U_{n,i}|,$$

and the inequality

$$0 \leq \sum_{i=1}^{2^{2n}} \frac{1}{2^n} |U_{n,i}| - \sum_{i=1}^{2^{2n}} \sum_{j=1}^{N_{n,i}} \frac{1}{2^n} |B^j_{n,i}| \leq 2^{2(n-1)} \varepsilon_n,$$
we conclude that
\[ \lim_n \mu_n(\mathbb{R}^{d+1}_+) = \lim_n \sum_{i=1}^{2^n} \sum_{j=1}^{N_{n,i}} (T_{n,i}^{j}d/2^n |B_{n,i}^{j}| = 1. \]

From this and (6.4) it follows that there is \( \mu \in \text{Prob}(\mathbb{R}^{d+1}_u,T) \), such that
\[ \lim_n \mu_n = \mu, \text{ narrowly in } \mathcal{M}_+(\mathbb{R}^{d+1}), \]
\[ \text{supp } \mu \subset \bar{P}, \mu(\bar{P}) = 1. \]

In particular \( \mu(\bar{P} \setminus P) = 0. \)

It remains to show that formula (2.1) holds with measure \( \mu \).

Let \( C_k \) be an increasing sequence of compact subset of \( P \), such that \( P = \cup_k C_k \).

The following property is implied by (6.4).

\( \mathcal{P}1: \text{ for any } \varepsilon > 0, \text{ there are } N \text{ and } K, \text{ such that } \mu_i((C_k)^c) = 0, \text{ for } i = 1..N, k > K, \text{ and } \]
\[ \sum_{i=N+1}^n \mu_i((C_k)^c) < \varepsilon, \quad n > N, k > K. \]

We will show now that
\[ \lim_k \lim_n \int_{(C_k)^c} \chi \left( \frac{|v-u|^2}{T} \right) \frac{1}{T^{d/2}} d\mu_n = 0, \text{ a.e. } v. \]

Indeed for any \( k \),
\[ \int_{(C_k)^c} \chi \left( \frac{|v-u|^2}{T} \right) \frac{1}{T^{d/2}} d\mu_n \text{ is increasing in } n, \text{ for any } v. \]

Moreover,
\[ \| \int_{(C_k)^c} \chi \left( \frac{|v-u|^2}{T} \right) \frac{1}{T^{d/2}} d\mu_n \|_{L^1_v} = \mu_n((C_k)^c). \]

Thus, by monotone convergence theorem,
\[ \| \lim_n \int_{(C_k)^c} \chi \left( \frac{|v-u|^2}{T} \right) \frac{1}{T^{d/2}} d\mu_n \|_{L^1_v} \leq \sup_n \mu_n((C_k)^c). \]

The right-hand side of this inequality converges to zero as \( k \to +\infty \), by property \( \mathcal{P}1 \).

We conclude that a.e. \( v \),
\[ \lim_k \int_{C_k} \chi \left( \frac{|v-u|^2}{T} \right) \frac{1}{T^{d/2}} d\mu = \lim_k \lim_n \int_{C_k} \chi \left( \frac{|v-u|^2}{T} \right) \frac{1}{T^{d/2}} d\mu_n \]
\[ = f(v) - \lim_k \lim_n \int_{(C_k)^c} \chi \left( \frac{|v-u|^2}{T} \right) \frac{1}{T^{d/2}} d\mu_n = f(v). \]

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