Variational properties of the kinetic solutions of scalar conservation laws

Misha Perepelitsa*

September 2, 2013

Abstract

We consider the variational kinetic formulation of the Cauchy problem for a scalar conservation law due to Brenier\[6\] and Panov[\]. The solutions in this formulation are represented by a kinetic density function $Y$ that solves a certain differential inclusion $\partial_t Y \in A(Y)$, in a suitable Hilbert space. In this paper we establish a sufficient “non-degeneracy” condition under which the operator $A$ is the maximal monotone operator. When this condition is satisfied, the theory of maximal monotone operators asserts that the solutions of the above differential inclusion are “slow” solutions, i.e., the solutions for which $\partial_t Y$ is the minimal norm element in the values of $A(Y)$. When the non-degeneracy condition doesn’t hold, the differential inclusion still has a unique solution, as was proved in Brenier[\]. We show that this solution is also a slow solution. We give an example of a kinetic density $Y$ containing a traveling wave discontinuity wave and show that $Y$ is the a slow solution when the traveling wave moves with the classical shock speed.

0.1 Introduction

We consider a scalar conservation law

$$\partial_t u + \nabla_x \cdot f(u) = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,$$

(1)

with the flux $f: \mathbb{R} \to \mathbb{R}^n$. A function $u(x,t)$ is called an entropy weak solution if for any convex entropy/entropy flux pairs $(\eta(u(x,t)), q(u(x,t)))$ of the flux function $f$,

$$\partial_t \eta + \nabla_x \cdot q \leq 0, \quad D'(\mathbb{R}^{n+1}).$$

(2)

The existence of unique entropy solutions for (1) with the initial data $u(t = 0) = u_0 \in L^\infty(\mathbb{R}^n)$, as well as their stability, was obtained in Kruzhkov[5]. The solution can be described using the kinetic formulation, as proved by Lions-Perthame-Tadmor[6]. In this approach, $u(x,t)$ is a weak entropy solution iff the kinetic density function

$$Y(x,t,v) = \begin{cases} -1 & u(x,t) < v < 0, \\ 1 & 0 < v < u(x,t), \\ 0 & \text{otherwise}, \end{cases}$$

(3)

*Department of Mathematics, University of Houston, Houston, Texas 77204-3008, misha@math.uh.edu
verifies the transport equation
\[
\partial_t Y + f_v(v) \cdot \nabla_x Y = \partial_v m, \quad D'(\mathbb{R}^{n+2}),
\]  
(4)

for some measure \( m \in \mathcal{M}_+(\mathbb{R}^{n+2}) \).

The kinetic formulation (4) can be expressed in a variational form which shows some interesting generalizations. To describe it, let us assume, for simplicity, that the initial data \( u_0(x) \), as well as the solution \( u(x,t) \) are non negative and periodic in \( x \) variable with the period \( \Pi = [0,1]^n \). Then, one can compute that in the sense of distributions,
\[
\partial_t \int_{\Pi \times \mathbb{R}_v} Y^2(x,t,v) \, dx \, dv = \partial_t \int_{\Pi} |u(x,t)| \, dx = 0.
\]  
(5)

The function \( Y \) is also an non-increasing function of \( v \) for each \( (x,t) \). Then, the condition (4) can be equivalently expressed (under suitable regularity assumptions on \( Y \)) via a variational form: for all non-decreasing in \( v \), periodic in \( x \), test functions \( \tilde{Y}(x,t,v) \),
\[
\int_{\Pi \times \mathbb{R}_v} (\tilde{Y} - Y)(\partial_t Y + f_v \cdot \nabla_x Y) \, dx \, dv \geq 0, \quad D'(\mathbb{R}_+).
\]  
(6)

Indeed, the equivalence holds because,
\[
\int_{\Pi \times \mathbb{R}_v} Y(\partial_t Y + f_v \cdot \nabla_x Y) \, dx \, dv = \partial_t \int_{\Pi \times \mathbb{R}_v} Y^2 \, dx \, dv = 0.
\]

The above mention generalization, due to Panov\[\] and also, independently, to Brenier, consists in allowing generic non-increasing in \( v \) functions \( Y(t,x,v) \), rather than functions with the values in the range in \( \{0,1\} \).

Before commenting on the advantages of this generalization, let us point out that now the variation kinetic formulation (6) is stronger than (4). The difference is that (6), in addition to (4), also requires \( \partial_t \int_{\Pi \times \mathbb{R}_v} Y^2 \, dx \, dv = 0 \). To see this, take \( \tilde{Y} = 2Y, \frac{1}{2}Y \) in (6).

A technical remark is in order here. By changing from \( Y \) to \( 1 - Y \), the kinetic formulations (4), (6) can be expressed using non-decreasing functions \( Y(t,x,v) \) and \( \tilde{Y} \) instead of non-increasing functions. Following the papers of Panov\[\], Brenier\[\], we will stick to the non-decreasing variant in our presentation.

The advantage of allowing general monotone functions is twofold. On one hand, to any solution \( Y \) of (6) one can associate a parametrized non-negative measure \( \nu_{x,t} = \partial_v Y \), which turns out to be a strong measure-valued solution of the conservation law (1), as was proved in Panov\[\]. In the same paper it was shown that for each initial data \( \nu_0 \in \mathcal{M}_+(\mathbb{R}_v) \) there is a unique, global strong measure-valued solution.

The set of strong measure-valued solutions is the subclass of the entropy measure-valued solutions that were introduced by Tartar\[10\] in his compensated compactness method. Such solutions were further studied by DiPerna\[4\], who showed that the weak entropy solutions are unique in the class of measure-valued solutions and by Schochet\[9\], who showed that entropy measure-valued solutions with a prescribed initial data are not unique.
One the other hand, the weak entropy solutions \(u(x, t)\) of (1) are “contained” in the graph of \(Y\) of the solution of (6); it was proved in Panov\] and, independently, by Brenier\], that for any \(\lambda \in \text{Int}(\text{Range}(Y))\),

\[
u_\lambda(x, t) = \sup \{ v : Y(t, x, v) \leq \lambda \},
\]

is a weak entropy solution of (1), and all solutions of (1) can be obtained in this way. In the paper Brenier\] the variational formulation (6) is expressed as a differential inclusion:

\[
\partial_t Y \in A(Y) := -f_v \cdot \nabla_x Y - \partial K(Y),
\]

where \(\partial K(Y)\) is the subdifferential to the indicator function of a convex, closed cone \(K\) consisting of all non-decreasing in \(v\) functions, in the space of \(L^2\) integrable functions, see section 0.2 for details. Additionally, Brenier\] proves the existence/uniqueness/stability of solutions and the regularity; if the initial data \(Y_0(x, v)\) is differentiable, \(\nabla_x Y_0 \in L^2_{x,v}\), then

\[
\partial_t Y, \nabla_x Y \in L^\infty((0, +\infty); L^2_{x,v}).
\]

It can also be shown that \(\partial_v Y \in L^\infty((0, +\infty); L^2_{x,v})\), if in addition \(\partial_t Y_0 \in L^2_{x,v}\).

The results [7, 8, 2] show a remarkable fact that all weak entropy solutions of (1) can be obtained through (7) (or (??)) from a globally stable and regular (if \(\partial_v Y_0 \in L^2_{x,v}\)) kinetic densities \(Y(t, x, v)\) – solutions of (8) (or (6)).

In this paper we further investigate the properties of solution of (8). The operator \(A(Y)\) is monotone but not necessarily maximal. In Lemma ?? we prove that under a certain non-degeneracy condition (??) on the characteristic velocities \(\partial_v f(v)\), \(A\) is a maximal monotone operator. Then, in Theorem ?? we prove the existence of the solutions of the Cauchy problem for (??)

The prove is done by approximating a Lipschitz continuous flux \(f(v)\) by a sequence of fluxes \(f_\varepsilon(v)\), satisfying the non-degeneracy condition (??).

The proof differs from the one in Brenier\]. Additionally, we show that solutions of (??) are slow solutions; for a.e. \(t > 0\),

\[
\|\partial_t Y(\cdot, t, \cdot)\| = \min_{V \in A(Y(\cdot, t, \cdot))} \|V\|.
\]

Alternatively, see theorem ??, this can be expressed \(\partial_t Y(\cdot, t, \cdot)\) minimizes the functional \(\|V + f_\varepsilon(v) \cdot \nabla_x Y(\cdot, t, \cdot)\|\) over all directions in the tangent cone \(V \in T_K(Y(t))\).

Such solutions can be considered as curves \(Y(t)\) with values in the admissible cone \(K\) for which the tangent \(\partial_t Y(t)\) belongs to the tangent cone \(T_K(Y(t)) = \text{closure}\{h(\tilde{Y} - Y(t)) : h > 0, \tilde{Y} \in K\}\).

After the prove of this result, which based on the fact that solutions of \(\partial_t Y \in -A(Y(t))\), with maximal monotone operator \(A\) are slow solutions, i.e., the solutions for which \(\|A(Y(t))\|\) is minimal, we show that there are traveling wave solution to (8) that correspond to the shock waves of (1). Such traveling waves move with the actual shock speed \(\sigma = \Delta f/\Delta u\). The shock speed appears in solving a minimization problem

\[
\min_{V \in T_K(Y(t))} \|V + f_v \cdot \nabla_x Y(t)\|_{L^2_{x,v}}.
\]
0.2 Main results

Let $\mathbb{H}$ be the space of $2L$-periodic in $x$, functions $Y(x,v)$ of $(x,v) \in \mathbb{R}^n \times [0,1]$, with the norm

$$||Y||^2 = \langle Y,Y \rangle = \int_{\mathbb{H} \times (0,1)} Y^2(x,v) \, dv \, dx,$$

Let $K \subset \mathbb{H}$ b a set of $Y$’s non-decreasing in $v$. $K$ is a closed cone and for any $Y \in K$, we denote

$$T_K(Y) = \mathbb{H} - \text{closure of } \{h(\bar{Y} - Y) : h \geq 0, \bar{Y} \in K\},$$

the tangent cone to $K$ at $Y$ and the normal cone (the subdifferential of $I_K(Y)$ — the indicator function of $K$):

$$\partial I_K(Y) = \{Z \in \mathbb{H} : \langle Z, \bar{Y} - Y \rangle \leq 0, \forall \bar{Y} \in K\}.$$  

We assume that $f \in C^1([0,1]^n)$, and denote by $a(v) = |\partial_v f(v)|$ and by $\sigma(v) = \partial_v f(v)/a(v)$, if $a > 0$.

Consider the Cauchy problem

$$\partial_t Y + \partial_v f \cdot \nabla_x Y \in -\partial I_K(Y),$$

$$Y(t = 0) = Y_0.$$  

The following theorem was proved in Brenier[2].

**Theorem 1.** (i.) Let $Y_0 \in K$, with $\nabla_x Y_0 \in \mathbb{H}$. Then there is a unique strong solution $Y \in C([0, +\infty); \mathbb{H} \cap K)$ of (11) with the following properties. $\{Y(\cdot,t,\cdot)\}_{t \geq 0}$ is nonexpansing semigroup on $\mathbb{H}$ and

$$\partial_t Y, \nabla_x Y \in L^\infty(0, +\infty; \mathbb{H}).$$

(ii.) If $Y_0 \in K$, there is a weak variational solution $Y \in C([0, +\infty); \mathbb{H} \cap K)$ which verifies (11) in the following sense.

As we mentioned in the introduction the proof of theorem 1 was based on using a time-discrete scheme that on each step was obtained by a free transport of values of $Y$ followed by a re-arrangement (collapse) in $v$ direction. We will obtain the result of theorem 1 by the means of the theory of differential inclusions with maximal monotone operators.

**Proof.** (Theorem 1) Note that part (ii.) follows by from part (i.) by considering solutions of (11) with $Y_{0,\varepsilon} \in K$, such that $\nabla_x Y_{0,\varepsilon} \in \mathbb{H}$, $Y_{0,\varepsilon} \to Y_0$ in $\mathbb{H}$, and passing to the limit in (11) on strong solutions $Y_{\varepsilon}$ corresponding to $Y_{0,\varepsilon}$.

To prove part (i.) we show first that $\partial_v f \cdot \nabla_x (\cdot) + \partial I_K(\cdot)$ is a maximal monotone operator (MMO) on $\mathbb{H}$ if $a(v) = |\partial_v f(v)|$ does not vanish.

Denote by $\mathcal{D}_\sigma$ the subset of $\mathbb{H}$ that consists of functions $f(x,v)$ that for a.a. $v \in (0,1)$ are weakly differentiable in the direction $\sigma(v)$ in $x$ variable, with $\partial_\sigma f \in \mathbb{H}$, where $\partial_\sigma$ is the derivative in the $\sigma(v)$ direction.

**Lemma 1.** Let $\inf_v a(v) > 0$. The operator $a(v)\partial_\sigma(\cdot) + \partial I_K(\cdot)$ is the maximal monotone operator with the domain $\mathcal{D}_\sigma \cap K$. 

4
Proof. We need to show that for any $W \in H$, there is a unique solution $Y \in D_\sigma \cap K$ of

$$Y + a\partial_\sigma Y + Z = W, \ Z \in \partial I_K(Y).$$

(13)

Uniqueness of the solution follows from the fact that $\partial I_K(\cdot)$ is MMO, $a\partial_\sigma(\cdot)$ is linear and

$$\int_{\Pi \times (0,1)} (a\partial_\sigma Y) Y \, dx dv = 0, \ Y \in D_\sigma.$$

To establish the existence of a solution we proceed in several steps.

**Step 1.** First we going to show that there is a solution of (13) for every $W \in H$ with $\nabla_x W \in H$. Let us approximate $\partial_\sigma Y$ by a finite difference:

$$\Delta_h Y(x,v) = \frac{1}{h} [Y(x + h\sigma(v)/2,v) - Y(x - h\sigma(v)/2,v)].$$

The operator $a(v)\Delta^h(\cdot)$ verifies the following properties for any $Y \in H$.

$$\int_{\Pi \times (0,1)} a\Delta^h(Y) Y \, dx dv = 0,$$

$$\int_{\Pi \times (0,1)} [a(v)\Delta^h(Y)(x + \varepsilon e_i, v) - a(v)\Delta^h(Y)(x, v)] \, dx dv = 0,$$

where $\varepsilon \in \mathbb{R}$ and $e_i$ $i^{th}$ coordinate unit vector. By the definition of the subdifferential of the indicator function of $K$, we obtain that

$$\int_{\Pi \times (0,1)} Z^h Y^h \, dx dv = 0,$$

Moreover $Z^h(\cdot + \varepsilon e_i, \cdot) \in \partial I_K(Y(\cdot + \varepsilon e_i, \cdot))$, and

$$\int_{\Pi \times (0,1)} (Z^h(x + \varepsilon e_i, v) - Z^h(x, v)) (Y^h(x + \varepsilon e_i, v) - Y^h(x, v)) \, dx dv \geq 0.$$

Operator $a(v)\Delta^h(\cdot)$ is Lipschitz continuous $H \to H$, for any $h \neq 0$. By lemma ?? of Brezis[3], operator $a\Delta^h(\cdot) + \partial I_K(\cdot)$ is MMO. Thus, there is a solution $Y^h$ of

$$Y^h + a\Delta^h Y^h + Z^h = W, \ Z^h \in \partial I_K(Y^h).$$

(14)

By using properties of $a\Delta^h$ and $Z^h$, stated above, one easily obtains that

$$\|Y^h\| \leq \|W\|,$$

(15)

$$\|Y^h(\cdot - \varepsilon e_i, \cdot) - Y^h(\cdot, \cdot)\| \leq \|W(\cdot - \varepsilon e_i, \cdot) - W(\cdot, \cdot)\|,$$

(16)

that hold for any $\varepsilon$ and $i = 1..n$. Taking $\varepsilon \to 0$, we conclude that $Y$ is weakly differentiable in $x$ and

$$\|\nabla_x Y^h\| \leq \|\nabla_x W\|.$$

We would like to show now that the set $\{Y^h\}_{h \in (0,1)}$ is relatively compact in $H$. 5
Claim 1. For any $\varepsilon \in (0, 1)$ there is a relatively compact set $\{Y^{h,\varepsilon}\}$ such that for any $h \in (0, 1)$,
\[\|Y^h - Y^{h,\varepsilon}\| \leq \varepsilon.\] (17)

Proof. Note that by the Sobolev embedding theorem, $Y^h$ belongs to $L^2(0, 1; L^p(\Pi))$ with $p \in [1, +\infty)$, if $n = 2$, and $p \in [1, 2n/(n - 2)]$, if $p > 2$, with the norm independent of $h$. For a.a. $v \in (0, 1)$ and $M > 0$, denote set
\[C^h_v(M) = \{x : Y^h(x, v) > M\}.
\]
Since
\[|C^h_v(M)| \leq M^{-2}\|Y^h(\cdot, v)\|^2,
\]
for any $p > 2$ we get
\[\int_0^1 |C^h_v(M)|^{(p-2)/p} dv \leq M^{-2(p-2)/p} \int_0^1 \|Y^h(\cdot, v)\|^{2(p-2)/p} dv \leq CM^{-a_0},\]
for some $a_0 > 0$ and $C$ - independent of $h$. Now, with any $p > 2$, if $n = 2$, or $p = 2n/(n-2)$, if $n > 2$, we obtain:
\[\int_{\{Y^h(\cdot) > M\}} (Y^h)^2 dxdv \leq CM^{-a_0} \int_0^1 \|Y^h(\cdot, v)\|^2_{L^p(\Pi)} dv \leq CM^{-a_0}.\] (18)
Define $Y^h(\cdot, v) = 0$ if $v < 0$ and set
\[Y^{h,\varepsilon}(x, v) = \frac{1}{\varepsilon} \int_0^\varepsilon Y^h(x, v - \bar{v}) d\bar{v},\]
for $(x, v) \in \Pi \times (0, 1)$. The new function $Y^{h,\varepsilon}$ is weakly differentiable in $(x, v)$ and for some $C(\varepsilon)$,
\[\|\nabla_{(x,v)} Y^{h,\varepsilon}\| \leq C(\varepsilon).
\]
It follows that for any $h > 0$, $\{Y^{h,\varepsilon}\}_{\varepsilon \in (0,1)}$ is relatively compact in $\mathbb{H}$. Using the monotonicity of $Y$ in $v$ and previous estimates we obtain:
\[\|Y^h - Y^{h,\varepsilon}\| = \int_{\Pi \times (0,1)} (Y^h - Y^{h,\varepsilon})^2 dxdv
\]
\[= \int_{\Pi \times (0,1)} \left(\frac{1}{\varepsilon} \int_0^\varepsilon Y^h(x, v) - Y^h(x, v - \bar{v}) d\bar{v}\right)^2 dxdv
\]
\[\leq \int_{\Pi \times (0,1)} (Y^h(x, v) - Y^h(x, v - \varepsilon))^2 dxdv
\]
\[\leq \int_{\{Y^h(\cdot) > M\}} (Y^h)^2 dxdv + M \int_{\Pi \times (0,1)} Y^h(x, v) - Y^h(x, v - \varepsilon) dxdv
\]
\[\leq CM^{-a_0} + M \int_{\Pi \times (0,1)} Y^h(x, v) dxdv \leq CM^{-a_0} + CM\varepsilon^{1/2}.
\]
Thus, (17) is verified. □
Now we continue the proof of lemma 1. On a suitable subsequence $h_k \to 0$, $Y^{h_k}$ converges to some $Y \in K$, with $\nabla_x Y \in \mathbb{H}$, and

i. $a\Delta^{h_k}(Y^{h_k})$ converges weakly in $\mathbb{H}$ to $a\partial_\sigma Y$;

ii. $Z^{h_k}$ converges weakly to some $Z \in \mathbb{H}$.

Since the operator $\partial I_K$ is strongly-weakly closed (see []), $Z \in \partial I_K(Y)$.

We proved that there is a solution of (13) for any $W$ with $\nabla_x W \in \mathbb{H}$.

**Step 2.** Let $W$ be an element of $\mathbb{H}$ and choose a sequence $W^{h_k} \to W$ in $\mathbb{H}$, with $\nabla_x W^{h_k} \in \mathbb{H}$.

Let $Y^{h_k}$ be the corresponding solution of (13). By monotonicity of $a\partial_\sigma(\cdot) + \partial I_K(\cdot)$,

$$\|Y^{h_1} - Y^{h_2}\| \leq \|W^{h_1} - W^{h_2}\|. $$

Thus $Y^{h_k}$ is a Cauchy sequence, convergent to some $Y \in \mathbb{H}$. We multiply (13) by $\partial_\sigma Y^{h_k}$ and use the identities

$$\int \int_{\Pi \times (0,1)} Y \partial_\sigma Y \, dx dv = 0, \quad \int \int_{\Pi \times (0,1)} Z \partial_\sigma Y \, dx dv = 0,$$

that hold for any $Y \in K$, $\nabla_x Y \in \mathbb{H}$, and $Z \in \partial I_K(Y)$, to obtain estimate on $\partial_\sigma Y^{h_k}$:

$$\|\partial_\sigma Y^{h_k}\| \leq C(\inf_v a(v))\|W^{h_k}\| \leq C,$$

for some $C$ independent of $h$. It follows then from the equation that

$$\|Z^{h_k}\| \leq C,$$

for some $C$ independent of $h$. Passing to the limit $h_k \to 0$, on a suitable subsequence, and using linearity of $\partial_\sigma(\cdot)$ and strong-weak closedness of $\partial I_K(\cdot)$, we obtain a solution $Y$ of (13). $\square$

Consider a Cauchy problem

$$\partial_t Y \in -a\partial_\sigma(Y) - \partial I_K(Y), \; Y(t = 0) = Y_0. \quad (19)$$

Under the non-degeneracy condition lemma 1, $a\partial_\sigma(\cdot) + \partial I_K(\cdot)$ is maximal monotone and the problem (19) has a unique solution with the properties listed in the next theorem; theorem 1, p.142 of Aubin-Celina[1].

**Theorem.** Let $Y_0 \in D_\sigma \cap K$. There is a unique solution $Y(t)$ of (19) for $t \in [0, +\infty)$, with the following properties: $Y(t) \in D_\sigma \cap K$;

$$Y \in C([0, T]; \mathbb{H}), \forall T > 0, \partial_\sigma Y, \partial_\sigma Y \in L^\infty(0, +\infty; \mathbb{H}).$$

Moreover,

i. $\{Y(t)\}_{t \geq 0}$ is a non-expansive semigroup on $\mathbb{H}$;

ii. if $\nabla_x Y_0 \in \mathbb{H}$, then for any $t > 0$, $\|\nabla_x Y(t)\| \leq \|\nabla_x Y_0\|;$

7
iii. $\partial_t Y(\cdot)$ is continuous from the right on $[0, +\infty)$ and $\|\partial_t Y(t)\| \leq \sup_v |\partial_v f(v)| \|\nabla_x Y_0\|$; iv. for any $t > 0$, $\partial_t Y$ is the element of the minimal norm in $-a\partial_\sigma Y - \partial I_K(Y)$.

Let $f \in C^1([0,1]^n)$ and $\{f_\varepsilon\}_{\varepsilon > 0}$ be a sequence $C^1$ functions such that: (i) $f_\varepsilon \to f$ in $C^1([0,1]^n)$; (ii) for all small $\varepsilon$, $\inf_v |\partial_v f_\varepsilon| > 0$. Such approximation clearly exist.

For each $f_\varepsilon$ and $Y_0 \in D_\sigma \cap K$, there is a solution $Y_\varepsilon$ that satisfies (19) with $f_\varepsilon$ and verifies the conclusions of the theorem cited above. It follows from the same theorem and assumptions on $f_\varepsilon$ that norms $\|Y_\varepsilon(t)\|$, $\|\partial_t Y_\varepsilon(t)\|$, $\|\nabla_x Y_\varepsilon(t)\|$ are uniformly bounded in $(t, \varepsilon) \in [0, +\infty) \times (0, \varepsilon_0)$. Moreover, by monotonicity we obtain:

$$\|Y_{\varepsilon_1}(t) - Y_{\varepsilon_2}(t)\| \leq \ell(\sup_v |\partial_v f_{\varepsilon_1} - \partial_v f_{\varepsilon_2}|) \|\nabla_x Y_0\|,$$

i.e. $\{Y_\varepsilon\}$ is relatively compact in $C([0, +\infty); \mathbb{H})$. With this and using the fact that $\partial I_K$, as a maximal monotone operator, is strongly-weakly closed, we obtain $Y = \lim Y_\varepsilon$ – the solution of (11) with $\partial_t Y, \nabla_x Y \in L^\infty(0, +\infty; \mathbb{H})$. This concludes the proof of theorem 1.

The theory of differential inclusions with MMO also implies that $Y(t)$ is the slow solution; see property (iv) of the above mentioned theorem. We recall this and its dual statements in the following corollary. For convenience of notation we denote by $Y(t) = Y(\cdot, t, \cdot) \in \mathbb{H}$.

**Corollary 1.** Let $Y_0 \in K$ and $\nabla_x Y_0 \in \mathbb{H}$. For the solution $Y \in C([0, +\infty); \mathbb{H})$ of (11), for all $t > 0$,

$$\|\partial_t Y(t)\| = \min_{Z \in \partial K(Y(t))} \|Z + \partial_v f \cdot \nabla_x Y(t)\|,$$

and

$$\|\partial_t Y(t) + \partial_v f \cdot \nabla_x Y(t)\| = \min_{V \in T_K(Y(t))} \|V + \partial_v f \cdot \nabla_x Y(t)\|.$$  

Moreover, the minimizers in (21), (22) are local in $x$: for all $t > 0$ and a.a. $x \in \Pi$,

$$\|\partial_t Y(x, t, \cdot)\|_{L^2(0,1)} = \min_{Z \in \partial I_K(Y(x, t, \cdot))} \|Z + \partial_v f \cdot \nabla_x Y(x, t, \cdot)\|_{L^2(0,1)},$$

and

$$\|\partial_t Y(x, t, \cdot) + \partial_v f \cdot \nabla_x Y(x, t, \cdot)\|_{L^2(0,1)} = \min_{V \in T_K(Y(x, t, \cdot))} \|V + \partial_v f \cdot \nabla_x Y(x, t, \cdot)\|_{L^2(0,1)},$$

where by $K$, $I_K$ and $T_K$ we denote the cone of nondecreasing functions in $L^2(0,1)$, its indicator function and the tangent cone, respectively.

**Proof.** Let us prove property (21). Consider operator $A(\cdot) = a\partial_\sigma (\cdot) + \partial I_K(\cdot)$. It is monotone and has a maximal extension $\tilde{A}$. Thus $Y$ is also a solution of a differential inclusion $\partial_t Y \in -\tilde{A}(Y)$, and thus (theorem ?? p. Aubin-Celina[?]), $\partial_t Y(t)$ is defined for all $t > 0$ and

$$\|\partial_t Y(t)\| = \min_{Z \in \tilde{A}(Y(t))} \|Z\|,$$
for all $t > 0$. Since also $\partial_t Y \in -A(Y)$, we arrive at (21).

Now we can prove (22). The tangent cone $T_K(Y(t))$, is defined by the condition that $V \in T_K(Y(t))$ if and only if $V \in \partial I_K(Y(t))$, $\langle V, Z \rangle \leq 0$. This means that $T_K(Y(t))$ is the polar cone to a convex closed cone $\partial I_K(Y(t))$. Property (iv.) of the theorem cited above states that

$$\partial_t Y = -a_{\partial \sigma} Y(t) - \pi_{\partial I_K}[-a_{\partial \sigma} Y(t)],$$

where $\pi_{\partial I_K}[]$ is the projector onto $\partial I_K(Y(t))$. This can be stated equivalently, that $\partial_t Y$ is the projection of $-a_{\partial \sigma} Y$ onto $T_K(Y(t))$, or

$$\|\partial_t Y(t) + a_{\partial \sigma} Y(t)\| = \min_{V \in T_K(Y(t))} \|V + a_{\partial \sigma} Y(t)\|,$$

for all $t > 0$.

To prove the local minimization properties (23) and (24) it is enough to notice that $Z \in \partial I_K(Y(t))$ in $\mathbb{H}$ if and only if $Z \in \mathbb{H}$ and for a.a. $x \in \Pi$, $Z(x, \cdot) \in \partial I_K(Y(x, t, \cdot))$ in the space $L^2(0, 1)$. Similarly, $V \in T_K(Y(t))$ in $\mathbb{H}$ if and only if $V \in \mathbb{H}$ and for a.a. $x \in \Pi$, $V(x, \cdot) \in T_K(Y(x, t, \cdot))$ in $L^2(0, 1)$.

Now we give a characterization of strong solutions of (19) that we will use in the next section.

**Lemma 2.** $Y \in C([0, T]; \mathbb{H} \cap K)$ with $Y(0) = Y_0$ and $\partial_t Y, -\nabla_x Y \in L^\infty(0, T; \mathbb{H})$, is a solution of the differential inclusion (11) (or (19)) if and only if

$$\int_{\Pi \times (0, +\infty) \times (0, 1)} Y \psi(v) \phi(x) \partial_t \tau(t) + Y \psi(v) \tau(t) \partial_v f \cdot \nabla_x \phi(x) \, dx \, dt \, dv \leq 0, \quad (25)$$

for any $\psi \in C^1([0, 1])$, $\partial_v \psi \geq 0$, $\phi(x) \in C^1(\Pi)$, $\Pi$ -periodic in $x$, $\tau \in C^1_0(0, +\infty)$ and

$$\|Y(t)\| = \|Y_0\|, \forall t > 0. \quad (26)$$

**Proof.** Given a solution of (19), (25) follows by multiplying (19) by $\psi \phi \tau$ and integrating by parts over $\Pi \times (0, +\infty) \times (0, 1)$. (26) follows by multiplying (19) by $Y$ and integrating in $(x, v)$.

Conversely, if $Y$ with the properties stated in lemma verifies (25) and (26), then differentiating in $t$ identity (26) squared to get

$$\int_{\Pi \times (0, 1)} (\partial_t Y) Y \, dx \, dv = 0, \text{ a.a. } t > 0,$$

and by periodicity,

$$\int_{\Pi \times (0, 1)} (\partial_v f(v) \cdot \nabla_x Y) Y \, dx \, dv = 0, \text{ a.a. } t > 0.$$

In (25), $(x, t)$ derivatives can be placed on $Y$ and together with previous two identities we obtain

$$\int_{\Pi \times (0, +\infty) \times (0, 1)} (\partial_t Y + \partial_v f(v) \cdot \nabla_x Y)(\psi(v) \phi(x) - Y) \tau(t) \, dx \, dt \, dv \leq 0,$$
for all test functions $\phi(x)\tau(t)\psi(v)$ as in (25). We conclude that

$$\partial_t Y + \partial_v f(v) \cdot \nabla_x Y \in -\partial I_K(Y), \text{ a.a } t > 0.$$  

\[ \square \]

### 0.3 An example

In this section we consider an example of a strong solution of (11) for which we explicitly compute the values of tangent vector $\partial_t Y$.

Consider a scalar conservation law (1) in one dimension with a convex flux function $f(u)$. We prescribe the initial data

$$u_0(x) = \begin{cases} u^+, & x \in [-L, 0] \cup [L/2, L], \\ u^-, & x \in (0, L/2), \end{cases}$$

with $u^+ > u^-, \; u^\pm \in (0, 1)$. The corresponding weak, entropy solution $u(x, t)$ of (1) consists (for small times) of a shock wave propagating from $x = 0$ with the speed

$$\sigma = (f(u^+) - f(u^-))/(u^+ - u^-)$$

and a rarefaction wave centred at $x = L/2$. The solution has this structure until the moment the shock wave collides with the r-wave. Let us choose a small $\varepsilon > 0$ and consider the kinetic formulation for this problem. We define

$$\tilde{Y}(x, t, v) = \begin{cases} 0, & v < u(x, t), \\ 1, & v \geq u(x, t). \end{cases}$$  

(27)

$Y(t, x, v)$ is the solution of (11) in the weak sense but it is not a strong solution: $\partial_t Y$ and $\nabla_x Y$ are in $H$. Let $T$ be so small that the shock and the rarefaction waves do not interact on $[0, T]$. We consider the regularization of $Y$,

$$Y_\varepsilon(x, t, v) = Y(x, t, v) * \omega_\varepsilon(x),$$

where $\omega_\varepsilon(x)$ is the standard (supported on $[x - \varepsilon, x + \varepsilon]$, non-negative, even, unit mass) smoothing kernel. $Y_\varepsilon$ verifies (25), because $Y$ does, as a kinetic function of an entropy solution. Moreover, $\partial_t Y_\varepsilon, \nabla_x Y_\varepsilon \in L^\infty(0, T; H)$. We claim that $Y_\varepsilon$ verifies condition (26) as well. Indeed, if $\varepsilon > 0$ is so small that the distance in $x$ direction between the shock wave and the left edge of the rarefaction wave is larger than $2\varepsilon$, then the difference

$$\int \Pi Y^2(x, t, v) \; dx - \int \Pi Y_\varepsilon^2(x, t, v) \; dx$$

is independent of $v \in [0, 1]$ (recall that $Y$ takes only values 0, 1). Since,

$$\int \int_{\Pi \times (0, 1)} Y^2(x, t, v) \; dx dv = \int \int_{\Pi \times (0, 1)} Y(x, t, v) \; dx dv$$

$$= \int \int_{\Pi \times (0, 1)} Y(x, 0, v) \; dx dv = \int \int_{\Pi \times (0, 1)} Y^2(x, 0, v) \; dx dv,$$
from the previous property of $Y_\varepsilon$ we conclude that
\[
\int_{\Pi \times (0,1)} Y_\varepsilon^2(x,t,v) \, dx dv = \int_{\Pi \times (0,1)} Y_\varepsilon^2(x,0,v) \, dx dv.
\]
Thus, by lemma 2, $Y_\varepsilon$ is the strong solution of (11)

Consider a cylinder $Q_T = [-\Delta, \Delta] \times [0,T]$, where $\Delta$ and $T$ are so chosen that $Q_T$ contains only the shock part of the weak solution $u(x,t)$. Since $u(x,t)$ (and $Y(x,t)$) is in the form a travelling wave on $Q_T$, $Y_\varepsilon$ is a travelling wave itself; i.e. $\partial_t Y_\varepsilon(x,t,v) = -\sigma \partial_x Y_\varepsilon(x,t,v)$, for any $v \in [0,1]$. By corollary 1, $\partial_t Y_\varepsilon$ is a minimizer of (24) (locally in $x$), and we obtain that the shock speed $\sigma$ appears in the minimization of the “interaction functional”:
\[
-\sigma \partial_x Y_\varepsilon(x,t,\cdot) = \arg \min_{V \in T\mathcal{K}(Y(x,t,\cdot))} \|V + \partial_v f(\cdot) \partial_x Y(x,t,\cdot)\|_{L^2(0,1)}.
\]

For $(x,t)$ corresponding to the rarefaction wave (a smooth solution of the conservation law) $\partial_t Y_\varepsilon = -\partial_v f \partial_x Y_\varepsilon$, because for smooth solutions, the kinetic function (for a fixed $v$) is trasported with the constant speed $\partial_v f(v)$.

References
