5.5 Rational Equations

Definition: Equations that contain at least one rational expression are called **rational equations.**

LCD stands for the "Least common denominator".

Solving rational equations

In this section, we will be solving rational equations. You'll know that you're doing one of these types of problems because the instructions that will tell you to "solve for *x*," or just to "solve."

To solve these types of equations, what you're *supposed* to do is to get a common denominator and one fraction on each side of the equal sign by doing the steps from section 5.3. Then once there's only one fraction on either side of the equal sign, you multiply by the least common denominator of either side and simplify. Then you solve for *x* using the algebra we learned earlier this semester.

Here's an example:

Solve $\frac{6}{x+4}+\frac{1}{x}=1$.

(1) Get a single fraction on both sides of "=": $\frac{6\cdot x}{(x+4)(x)}+\frac{1(x+4)}{x(x+4)}=\frac{1}{1}\rightarrow\frac{7x+4}{x(x+4)}=\frac{1}{1}$

(2) Multiply by LCD of both denominators and simplify: $\frac{\cancel{x}(x+4)}{1}\cdot\frac{7x+4}{\cancel{x}(x+4)}=\frac{x(x+4)}{1}\cdot\frac{1}{1}$

(3) Solve using algebra:

$$7x + 4 = x^2 + 4x$$
$$x^2 - 3x - 4 = 0$$
$$(x - 4)(x + 1) = 0$$
$$x - 4 = 0 \text{ or } x + 1 = 0,$$
$$\text{so } x = 4, \text{ or } x = -1$$

This is the longer, tedious *official* process to solve rational equations. There is a shorter method, though. You don't have to do quite so many steps to solve these equations. The trick is to find the LCD of all the fractions in the problem (if a piece of your problem isn't a fraction, make it into one by putting it over 1), and then multiply every piece of the problem by that LCD, simplify and solve. What makes this better? You get to skip step (1) in the process above.

Here's the same problem from before, but now using the trick: $\frac{6}{x+4}+\frac{1}{x}=1$

(1) Find LCD of both sides of problem: $x(x + 4)(1) = x(x + 4)$

(2) Multiply each piece by the LCD & simplify: $\frac{x(x+4)}{1}\cdot\frac{6}{x+4}+\frac{\cancel{x}(x+4)}{1}\cdot\frac{1}{\cancel{x}}=\frac{x(x+4)}{1}\cdot\frac{1}{1}$

(3) Rewrite and solve using algebra: $6x + x + 4 = x^2 + 4x$

$$7x + 4 = x^2 + 4x$$
$$x^2 - 3x - 4 = 0$$
$$(x - 4)(x + 1) = 0$$

$$x - 4 = 0 \text{ or } x + 1 = 0,$$
$$\text{so } x = 4, \text{ or } x = -1$$

See? You do easier steps, but you get the same answers. Since it's easier to do this second method, we'll be doing that method to solve these types of problems.

Extraneous Solutions
There's one last thing that you have to check before you're done with the problem.

Definition: A **restriction** on a rational expression is the value that would make the denominator equal to zero.
Definition: An **extraneous solution** is a value that is obtained from solving an equation that does not satisfy the original equation.

You have to check that the solutions you get don't include any numbers that are the restrictions on the rational expressions in the problem. Here's an example of why you need to check:

$$\frac{6}{2x-6} = \frac{x}{x-3} - \frac{3}{2}$$

(1) Find the LCD of both sides of the problem: $2(x - 3)$
(2) Multiply each piece by LCD & simplify: $\frac{\cancel{2}(x-3)}{1} \cdot \frac{6}{\cancel{2}(x-3)} = \frac{2(x-3)}{1} \cdot \frac{x}{x-3} - \frac{\cancel{2}(x-3)}{1} \cdot \frac{3}{\cancel{2}}$
(3) Rewrite and solve using algebra: $6 = 2x - 3x - 9$

$$6 = -x - 9$$
$$x = 3$$

So, it looks like the answer should be 3, but take a look at what the restrictions are for this formula. On the left-hand side of the equal sign, notice that you have $2x - 6$. If you plug the solution into the left-hand side, you'll have a zero in the denominator. You can't have the solution to the problem make the denominator of any of the fractions equal to zero. So we say that this problem has *no solution* or an *extraneous solution at* $x = 3$. Once you finish solving the problem, just check that the solution you've gotten doesn't make any of the denominators equal to zero when you plug in the number(s).

Examples:

1. Solve : $\dfrac{x-2}{2x+4}=-2$

2. Solve: $\dfrac{x+5}{x^2+4x+1}=1$

3. Solve: $\dfrac{2x}{3}+\dfrac{5x}{6}=\dfrac{1}{3}$

4. Solve: $\dfrac{3}{8}+\dfrac{4}{x+1}=6$

5. Solve: $\dfrac{1}{5x}+\dfrac{2}{9x}=-1$

6. Solve: $4-\frac{2}{x}=\frac{1}{2}$

7. Solve: $\frac{3}{x-4}-\frac{6}{x+4}=\frac{24}{x^2-16}$

8. Solve: $\frac{3}{x+4}-\frac{6}{7}=\frac{2}{x+4}$

9. Solve: $\frac{5}{x^2}+\frac{4}{x}=1$

10. Solve: $\frac{5}{x+2}+\frac{4}{x-5}=\frac{1}{x^2-3x-10}$