Math 1314
Lesson 23
Partial Derivatives

When we are asked to find the derivative of a function of a single variable, $f(x)$, we know exactly what to do. However, when we have a function of two variables, there is some ambiguity. With a function of two variables, we can find the slope of the tangent line at a point P from an infinite number of directions. We will only consider two directions, either parallel to the x axis or parallel to the y axis. When we do this, we fix one of the variables. Then we can find the derivative with respect to the other variable.

Treat as a constant

So, if we fix y, we can find the derivative of the function with respect to the variable x. And if we fix x, we can find the derivative of the function with respect to the variable y.

These derivatives are called **partial derivatives**.

w.r.t. = with respect to

First-Order Partial Derivatives

We will use two different notations:

$\frac{\partial f}{\partial x} = f_x$ In this case, you consider y as a constant. Derivative w.r.t. x

$\frac{\partial f}{\partial y} = f_y$ In this case, you consider x as a constant. Derivative wrt y

We can use GGB to determine the first-order partial derivatives. The command is:
derivative[<function>,<variable>]

Example 1: Suppose $f(x, y) = x^2 - 3xy^2 + 4y^2$. *Enter the function into GGB.*

a. Find $\frac{\partial f}{\partial x} = f_x$.

Command: derivative[f,x]

Answer: $f_x = a(x,y) = 2x - 3y^2$

b. Find $\frac{\partial f}{\partial y} = f_y$.

Command: derivative[f,y]

Answer: $f_y = b(x,y) = -6xy + 8y$

We can also evaluate the first partial derivatives at a given point.

Example 2: Find the first-order partial derivatives of the function $f(x,y)=4x^3y^2+2x^2y^3-12x^2+3y^2+10$ evaluated at the point (-1, 3).
Enter the function into GGB.

a. $f_x\big|_{(-1,3)}$

Command: derivative[f, x]

Answer: $f_x = a(x,y) = 12x^2y^2 + 4xy^3 - 24x$

Command: a(-1,3)

Answer: 24

b. $f_y\big|_{(-1,3)}$

Command: derivative[f, y]

Answer: $f_y = b(x,y) = 8x^3 + 6x^2y^2 + 6y$

Command: b(-1,3)

Answer: 48

Second-Order Partial Derivatives

Sometimes we will need to find the second-order partial derivatives. To find a second-order partial derivative, you will take respective partial derivatives of the first partial derivative. There are a total of 4 second-order partial derivatives.

There are two notations, but we will only use one of them.

1st & 2nd x

$$f_{xx}=\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

1st x 2nd y

$$f_{xy}=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

1st & 2nd y

$$f_{yy}=\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

1st y 2nd x

$$f_{yx}=\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

Example 3: Evaluate the second-order partial derivatives of $f(x,y)=3x^2-x^3y^3+5xy+6y^3$ at the point (1, 2). *Enter the function into GGB. Then produce the first-order partials.*

a. $f_{xx}\big|_{(1,2)}$

Command:

1st order derivative[f, x]

2nd order derivative[a, x]

Answer:

$f_x = a(x,y) = 6x - 3x^2y^3 + 5y$

$f_{xx} = b(x,y) = 6 - 6xy^3$

Command:

b(1, 2)

Answer:

= −42

b. $f_{xy}\big|_{(1,2)}$

Command:

2nd order derivative[a, y]

Answer:

$d(x,y) = -9x^2y^2 + 5$

Command:

d(1, 2)

Answer:

e = −31

c. $f_{yy}\big|_{(1,2)}$

Command:

1st order derivative[f, y]

2nd order derivative[g, y]

Answer:

$f_y = g(x,y) = -3x^3y^2 + 5x + 18y^2$

$f_{yy} = h(x,y) = -6x^3y + 36y$

Command:

h(1, 2)

Answer:

i = 60

d. $f_{yx}\big|_{(1,2)}$

Command:

2nd order derivative[g, x]

Answer:

$f_{yx} = j(x,y) = -9x^2y^2 + 5$

Command:

j(1, 2)

Answer:

k = −31

A function of the form $f(x, y) = ax^b y^{1-b}$ where a and b are positive constants and $0 < b < 1$ is called a ***Cobb-Douglas production function***. In this function, x represents the amount of money spent for labor, and y represents the amount of money spent on capital expenditures such as factories, equipment, machinery, tools, etc. The function measures the output of finished products.

The first partial with respect to x is called the **marginal productivity of labor**. It measures the change in productivity with respect to the amount of money spent for labor. In finding the first partial with respect to x, the amount of money spent on capital is held at a constant level.

The first partial with respect to y is called the **marginal productivity of capital**. It measures the change in productivity with respect to the amount of money spent on capital expenditures. In finding the first partial with respect to y, the amount of money spent on labor is held at a constant level.

Example 4: A country's production can be modeled by the function $f(x, y) = 50x^{2/3}y^{1/3}$ where x gives the units of labor that are used and y represents the units of capital that were used.

a. Find the first-order partial derivatives and label each as marginal productivity of labor or marginal productivity of capital.
Enter the function into GGB.

Command: derivative[f, x]

Answer: $f_x = a(x,y) = \dfrac{100\, y^{1/3}}{3\, x^{1/3}}$ Marginal productivity of labor

Command: derivative[f, y]

Answer: $f_y = b(x,y) = \dfrac{50\, x^{2/3}}{3\, y^{2/3}}$ Marginal productivity of capital

b. Find the marginal productivity of labor and the marginal productivity of capital when the amount expended on labor is 125 units and the amount spent on capital is 27 units.

Command: a(125, 27)

Answer: 20 units / units of increase in labor

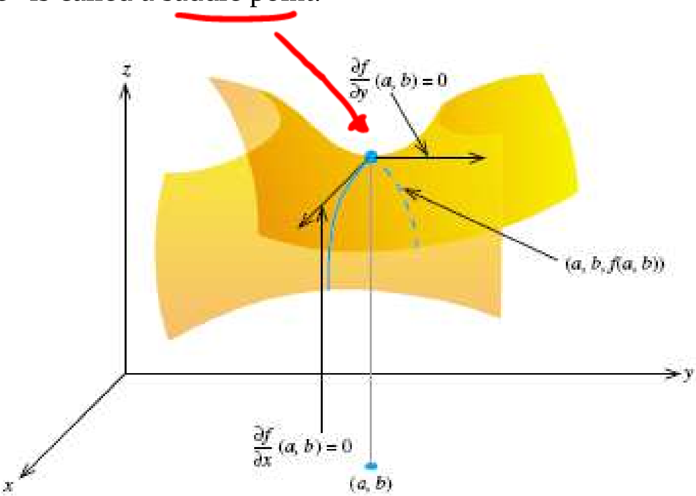
Command: b(125, 27)

Answer: 46.3 units / units of increase in Capital

Math 1314
Lesson 24
Maxima and Minima of Functions of Several Variables

We learned to find the maxima and minima of a function of a single variable earlier in the course. We had a second derivative test to determine whether a critical point of a function of a single variable generated a maximum or a minimum, or possibly that the test was not conclusive at that point. We will use a similar technique to find relative extrema of a function of several variables.

Since the graphs of these functions are more complicated, determining relative extrema is also more complicated. At a specific critical number, we can have a max, a min, or something else. That "something else" is called a saddle point.



The method for finding relative extrema is very similar to what you did earlier in the course.

1. Find the first partial derivatives and set them equal to zero. You will have a system of equations in two variables which you will need to solve to find the critical points.

2. Apply the second derivative test. To do this, you must find the second-order partial derivatives. Let $D(x, y) = f_{xx} f_{yy} - (f_{xy})^2$. You will compute $D(a,b)$ for each critical point (a,b). Then you can apply the second derivative test for functions of two variables:

- If $D(a, b) > 0$ and $f_{xx}(a,b) < 0$, then f has a relative maximum at (a, b).
- If $D(a, b) > 0$ and $f_{xx}(a,b) > 0$, then f has a relative minimum at (a, b).
- If $D(a, b) < 0$, then f has neither a relative maximum nor a relative minimum at (a, b) (i.e., it has a saddle point, which is neither a max nor a min).
- If $D(a, b) = 0$, then this test is inconclusive.

Example 1: Find the relative extrema of the function. $f(x,y) = -3x^2 + 2xy - 2y^2 + 14x + 2y - 8$

Begin by entering the function into GGB.

a. Find the first-order partials.

Command: derivative[f, x]

Answer: $f_x = a(x,y) = -6x + 2y + 14$

Command: derivative[f, y]

Answer: $f_y = b(x,y) = 2x - 4y + 2$

b. Set each first-order partial equal to zero and enter each into GGB.

Command: $-6x + 2y + 14 = 0$

Answer: $c: -3x + y = -7$

Command: $2x - 4y + 2 = 0$

Answer: $d: x - 2y = -1$

c. Find the point of intersection of the equations in part B. These points of intersection are the critical points of the function *f*.

Command: intersect[c, d]

Answer: $(3, 2)$ ← critical point

d. Determine $D(x,y) = f_{xx} \cdot f_{yy} - \left(f_{xy}\right)^2$.

Command: derivative[a, x]

Answer: $f_{xx} = g(x,y) = -6$

Command: derivative[b, y]

Answer: $f_{yy} = h(x,y) = -4$

Command: derivative[a, y]

Answer: $f_{xy} = i(x,y) = 2$

e. Apply the second derivative test to classify each critical point found in step C.

$D(x,y) = (-6)(-4) - (2)^2$

$= 20 > 0$

$f_{xx} = -6 < 0$

← R. Max at $(3,2)$

f. For any maxima point and minima point found in part E, calculate the maxima and minima, respectively.

Command: $f(3, 2)$

Answer: 3

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Example 2: Find the relative extrema of the function $f(x, y) = 2x^3 + y^2 - 9x^2 - 4y + 12x - 2$.
Begin by entering the function into GGB.
a. Find the first-order partials.
Command: Answer:

Command: Answer:

b. Set each first-order partial equal to zero and enter each into GGB.
Command: Answer:

Command: Answer:

c. Find the point of intersection of the equations in part B. These points of intersection are the critical points of the function *f*.
Command: Answer:

d. Determine $D(x, y) = f_{xx} \cdot f_{yy} - \left(f_{xy} \right)^2$.
Command: Answer:

Command: Answer:

Command: Answer:

e. Apply the second derivative test to classify each critical point found in step C.

f. For any maxima point and minima point found in part E, calculate the maxima and minima, respectively.
Command: Answer:

Example 3: Suppose a company's weekly profits can be modeled by the function $P(x, y) = -0.2x^2 - 0.25y^2 - 0.2xy + 100x + 90y - 4000$ where profits are given in thousand dollars and x and y denote the number of standard items and the number of deluxe items, respectively, that the company will produce and sell. How many of each type of item should be manufactured each week to maximize profit? What is the maximum profit that is realizable in this situation?

Begin by entering the function into GGB.

a. Find the first-order partials.

Command: Answer:

Command: Answer:

b. Set each first-order partial equal to zero and enter each into GGB.

Command: Answer:

Command: Answer:

c. Find the point of intersection of the equations in part B. These points of intersection are the critical points of the function *f*.

Command: Answer:

d. Determine $D(x, y) = f_{xx} \cdot f_{yy} - \left(f_{xy} \right)^2$.

Command: Answer:

Command: Answer:

Command: Answer:

e. Apply the second derivative test to classify each critical point found in step C.

f. How many of each type of item should be manufactured each week to maximize profit?

g. What is the maximum profit that is realizable in this situation?
Command: Answer: