Lesson 18 – Finding Indefinite and Definite Integrals

Math 1314
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Working with Riemann sums can be quite time consuming, and at best we get a good approximation. In an area problem, we want an exact area, not an approximation. The definite integral will give us the exact area, so we need to see how we can find this.

We need to start by finding an antiderivative:

**Antiderivatives (Indefinite Integrals)**

**Definition:** A function \( F \) is an antiderivative of \( f \) on interval \( I \) if \( F'(x) = f(x) \) for all \( x \) in \( I \).

You can think of an antiderivative problem as asking you to find the problem if you are given the derivative. Look at what you are given in your problem and ask: “If this is the answer, what was the problem whose derivative I wanted to find?”

Antiderivative problems will use this notation: \( \int f(x) \, dx = F(x) + C \), \( C \) is an arbitrary constant

**Notation:** We will use the integral sign \( \int \) to indicate integration (antidifferentiation). This indicates that the indefinite integral of \( f(x) \) with respect to the variable \( x \) is \( F(x) + C \) where \( F(x) \) is an antiderivative of \( f \).

The reason for “+ \( C \)” is illustrated below:

Each function that follows is an antiderivative of \( 10x \) since the derivative of each is \( 10x \).

\( F(x) = 5x^2 - 1 \), \( G(x) = 5x^2 + 1 \), \( H(x) = 5x^2 + 2 \), \( K(x) = 5x^2 + 2 \), etc.

**Basic Rules**

**Rule 1:** The Indefinite Integral of a Constant \( k \)

\[ \int k \, dx = kx + C \]

**Example 1:**

A. \( \int -10 \, dx \)

B. \( \int \sqrt{5} \, dx \)
Rule 2: The Power Rule

\[ \int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \ n \text{ is a real number with } n \neq -1 \]

Example 2: \[ \int x^4 \, dx \]

Example 3: \[ \int \sqrt{x} \, dx \]

Example 4: \[ \int \frac{1}{x} \, dx \]
Rule 3: The Indefinite Integral of a Constant Multiple of a Function

\[ \int cf(x)\,dx = c \int f(x)\,dx \]

**Example 5:** \[ \int -5x^9\,dx \]

**Example 6:** \[ \int \frac{2}{x^3}\,dx \]

**Example 7:** \[ \int -\frac{4}{\sqrt[3]{x^2}}\,dx \]
Rule 4: The Sum (Difference) Rule

\[ \int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx \]

Example 8: \( \int (2x^2 - 10x - 1) dx \)

Example 9: \( \int (11x^{10} - 4x^9) dx \)

The Fundamental Theorem of Calculus

Finding the antiderivative is a tool that we need in order to find the definite integral of a function over an interval. Next we apply the fundamental theorem of calculus:

Let \( f \) be a continuous function on \([a, b]\). Then \( \int_a^b f(x) dx = F(b) - F(a) \) where \( F(x) \) is any antiderivative of \( f \).

This says that we can find the definite integral by first finding the antiderivative of the function that’s given and then by evaluating the antiderivative at the upper and lower limits of integration and subtracting.

If the function is non-negative (never dips below the x-axis) then the definite integral gives the area under the curve on the interval \([a, b]\). If the function crosses the x axis, so that some of its y values are below the x-axis, then the definite integral gives the “net” of the two areas. Subtract the area of the part that is below the axis from the area of the part that is above the axis. If the area below the axis is larger, it is possible to get a definite integral that is negative.
Properties of Definite Integrals

Suppose \( f(x) \) and \( g(x) \) are integrable functions. Then:

1. \( \int_{a}^{a} f(x) \, dx = 0 \)
2. \( \int_{a}^{b} cf(x) \, dx = c \int_{a}^{b} f(x) \, dx \)
3. \( \int_{a}^{b} f(x) \, dx = -\int_{b}^{a} f(x) \, dx \)
4. \( \int_{a}^{b} [f(x) \pm g(x)] \, dx = \int_{a}^{a} f(x) \, dx \pm \int_{a}^{b} g(x) \, dx \)
5. \( \int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx \), \( a < c < b \)

**Example 10:** Evaluate: \( \int_{-1}^{5} 3(x^2 - 2) \, dx \)

**Example 11:** Evaluate: \( \int_{1}^{2} \left( x - \frac{3}{\sqrt{x}} \right) \, dx \)
Example 12: Suppose \( f(x) = 2x + 1 \). Find the area under the graph of \( f \) and above the \( x \) axis from \( x = -2 \) to \( x = 10 \).

Example 13: Let \( \int_{0}^{3} f(x) \, dx = 4 \), \( \int_{0}^{3} [f(x)]^2 \, dx = 16 \) and \( \int_{0}^{3} kd\, x = 2 \). Find \( \int_{0}^{3} \left( 3[f(x)]^2 + 2f(x) - 5k \right) \, dx \).

Example 14: Let \( \int_{0}^{2} f(x) \, dx = 4 \), \( \int_{0}^{4} f(x) \, dx = 5 \) and \( \int_{4}^{7} f(x) \, dx = 4 \). Find \( \int_{2}^{7} f(x) \, dx \).

Recall: \( \int_{a}^{b} f(x) \, dx = -\int_{b}^{a} f(x) \, dx \)
Example 15: The graph of $f$ is shown below. In the graph, A, B, C, and D are the areas of the regions given on the graph. Region A = 20, B = 14, C = 6 and D = 25. The graph may not be drawn to scale.

Find $\int_{-4}^{4} f(x)dx$