Lesson 5 – One-sided Limits and Continuity

Sometimes we are only interested in the behavior of a function when we look from one side and not from the other.

Example 1: Use the given graph to find \( \lim_{x \to 0} f(x) \).

Now suppose we are only interested in looking at the values of \( x \) that are bigger than 0. In this case, we are looking at a one-sided limit. We write \( \lim_{x \to 0^+} f(x) \). This is called a right-hand limit, because we are looking at values on the right side of the target number.

In this case, \( \lim_{x \to 0^+} f(x) = -1 \).

If we are interested in looking only at the values of \( x \) that are smaller than 0, then we would be finding the left-hand limit. The values of \( x \) that are smaller than 0 are to the left of 0 on the number line, hence the name. We write \( \lim_{x \to 0^-} f(x) \).

In this case, \( \lim_{x \to 0^-} f(x) = 1 \).

Our definition of a limit from the last lesson is consistent with this information. We say that \( \lim_{x \to a} f(x) = L \), if and only if the function approaches the same value, \( L \), from both the left side and the right side of the target number. This idea is formalized in this theorem:

**Theorem:** Let \( f \) be a function that is defined for all values of \( x \) close to the target number \( a \), except perhaps at \( a \) itself. Then \( \lim_{x \to a} f(x) = L \) if and only if \( \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L \).

We can also find one-sided limits from piecewise defined functions.
Example 2: Consider this graph:

![Graph Image]

Find each of the following limits, if it exist.
A. \( \lim_{x \to 0^-} f(x) \)  
B. \( \lim_{x \to 0^+} f(x) \)  
C. \( \lim_{x \to 0} f(x) \)

Example 3: Suppose \( f(x) = \begin{cases} 
  x^2 - x + 2, & x < 1 \\
  x + 1, & 1 \leq x < 2 \\
  x^3 - 4, & x \geq 2 
\end{cases} \). Find each of the following limits, if it exist.

A. \( \lim_{x \to 2^-} f(x) \)  
B. \( \lim_{x \to 2^+} f(x) \)  
C. \( \lim_{x \to 2} f(x) \)

D. \( \lim_{x \to 1^-} f(x) \)  
E. \( \lim_{x \to 1^+} f(x) \)  
F. \( \lim_{x \to 1} f(x) \)
Continuity at a Point

Here’s the general idea of continuity at a point: a function is continuous at a point if its graph has no gaps, holes, breaks or jumps at that point. Stated a bit more formally,

A function \( f \) is said to be continuous at the point \( x = a \) if the following three conditions are met:

1. \( f(a) \) is defined
2. \( \lim_{{x \to a}} f(x) \) exists
3. \( \lim_{{x \to a}} f(x) = f(a) \)

If a function is not continuous at a point, then we say it is discontinuous at that point.

We find points of discontinuity by examining the function that we are given. A function can have a removable discontinuity, a jump discontinuity or an infinite discontinuity.

Example 4: The graph of a function given below is discontinuous at some values of \( x \). State the \( x \)-values of where the function is discontinuous then state why the function is discontinuous at each one of those points.
Let \( f(x) \) be discontinuous at \( x = a \). Then:

- If \( \lim_{{x \to a^-}} f(x) \) and \( \lim_{{x \to a^+}} f(x) \) exists, but are not equal (i.e. \( \lim_{{x \to a}} f(x) \) does not exist) then at \( a \) there is a **jump discontinuity**.

- If \( \lim_{{x \to a^-}} f(x) \) exists but \( \lim_{{x \to a^+}} f(x) \neq f(a) \), then at \( a \) there is a **removable discontinuity**.

- If \( \lim_{{x \to a^-}} f(x) \) and/or \( \lim_{{x \to a^+}} f(x) \) is/are infinite, then at \( a \) there is an **infinite discontinuity**.

**Example 5:** Let \( f(x) = \begin{cases} x - 6, & x \leq 0 \\ x^2 + 5x + 6, & x > 0 \end{cases} \) is the function continuous at \( x = 0 \)?

We need to check:

1. Is \( f(0) \) defined?

2. Does \( \lim_{{x \to 0}} f(x) \) exist?

3. \( \lim_{{x \to 0}} f(x) = f(0) \)?
Example 6: Let \( f(x) = \begin{cases} \frac{x^2 - 25}{5 + x}, & x \neq -5 \\ -10, & x = -5 \end{cases} \) is the function continuous at \( x = -5 \)?

We need to check:

1. Is \( f(-5) \) defined?

2. Does \( \lim_{x \to -5} f(x) \) exist?

3. \( \lim_{x \to -5} f(x) = f(-5) \)?
Continuity

We will be interested in finding where a function is continuous and where it is discontinuous. We’ll look at continuity over the entire domain of the function, over a given interval and at a specific point.

Continuity over an Interval

A function is continuous over the interval \((a, b)\) if it is continuous at every point in the interval. We’ll state answers using interval notation.

Example 7: Find the intervals on which \(f\) is continuous:

a. \(f(x) = \frac{7 - x}{x^2 - 5x - 14}\)

b. \(f(x) = 3x^4 - 5x^2 + 2x - 7\)

An Application Involving Limits

Example 8: The average cost in dollars of constructing each skateboard when \(x\) skateboards are produced can be modeled by the function \(\bar{C}(x) = 12.5 + \frac{123,500}{x}\). What is the average cost per skateboard if the number of boards produced gets larger?