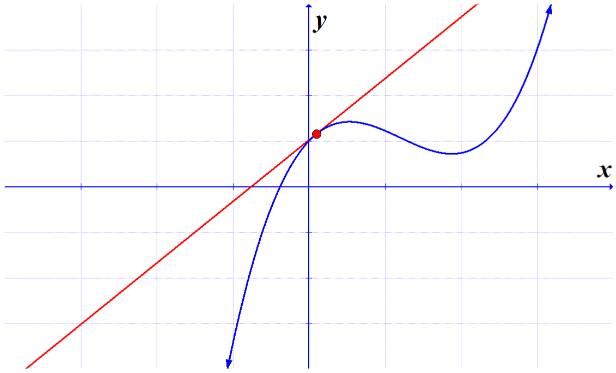
Math 1314
Lesson 6
The Limit Definition of the Derivative; Rules for Finding Derivatives

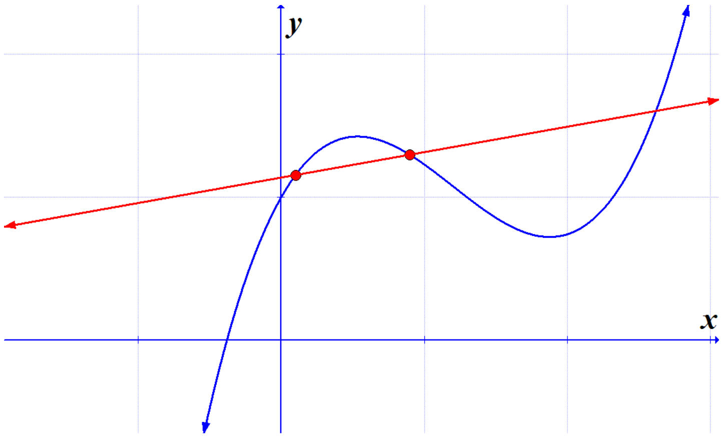
We now address the first of the two questions of calculus, the tangent line question.

We are interested in finding the slope of the tangent line at a specific point.



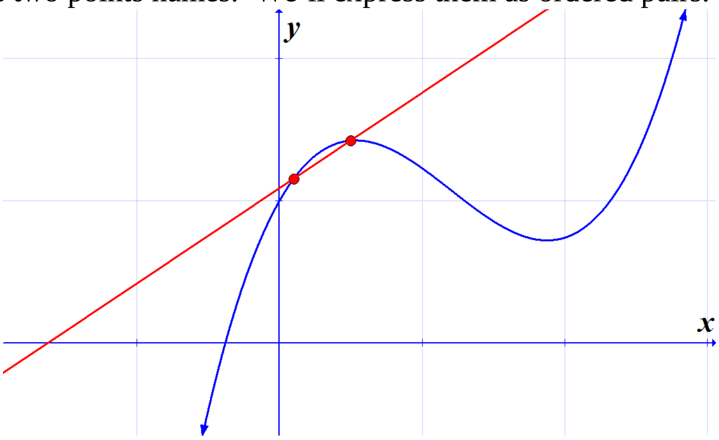
We need a way to find the slope of the tangent line analytically for every problem that will be exact every time.

We can draw a secant line across the curve, then take the coordinates of the two points on the curve, *P* and *Q*, and use the slope formula to approximate the slope of the tangent line.



Now suppose we move point *Q* closer to point *P*. When we do this, we'll get a better approximation of the slope of the tangent line. When we continue to move point *Q* even closer to point *P*, we get an even better approximation. We are letting the distance between *P* and *Q* get smaller and smaller.

Now let's give these two points names. We'll express them as ordered pairs.



Now we'll apply the slope formula to these two points.

$$m = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

This expression is called a **difference quotient** also called the **average rate of change.**

The last thing that we want to do is to let the distance between *P* and *Q* get arbitrarily small, so we'll take a limit.

This gives us the definition of the **slope of the tangent line**.

The slope of the tangent line to the graph of f at the point $P(x, f(x))$ is given by

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists.

We find the **instantaneous rate of change** when we take the limit of the difference quotient.

The **derivative of *f* with respect to *x*** is the function f' (read "*f* prime") defined by

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. The domain of $f'(x)$ is the set of all *x* for which the limit exists.

We can use the derivative of a function to solve many types of problems.   But first we need a method for finding the derivative.

Example 1: Suppose the distance covered by a car can be measured by the function $s(t) = 4t^2 + 32t$, where $s(t)$ is given in feet and t is measured in seconds. Find the average velocity of the car over the interval [0, 4].

- *Recall that "velocity" is the same as "rate of change", and so "average velocity" is the same as the "average rate of change"*
- *Also recall that the "average rate of change" is the same as the difference quotient,* $\dfrac{f(x+h)-f(x)}{h}$.

The Four-Step Process for Finding the Derivative

Now that we know what the derivative is, we need to be able to find the derivative of a function. We'll use an algebraic process to do so. We'll use a Four-Step Process to find the derivative. The steps are as follows:

1. Find $f(x+h)$.
2. Find $f(x+h)-f(x)$.
3. Form the difference quotient $\dfrac{f(x+h)-f(x)}{h}$.
4. Find the limit of the difference quotient as h gets close to 0: $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

So, $f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$.

Example 2: Use the four-step process to find the derivative of $f(x) = x^2 + 4x$.

Step 1: $f(x+h)$.

Step 2: $f(x+h)-f(x)$.

Step 3: $\dfrac{f(x+h)-f(x)}{h}$.

Step 4: $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

Now find the instantaneous rate of change of y at x = -1. (Recall that the **instantaneous rate of change** is when we take the limit of the difference quotient.)

Rules for Finding Derivatives

We can use the limit definition of the derivative to find the derivative of every function, but it isn't always convenient. Fortunately, there are some rules for finding derivatives which will make this easier.

First, a bit of notation: $\dfrac{d}{dx}\left[f(x)\right]$ is a notation that means "the derivative of *f* with respect to *x*, evaluated at *x*."

Rule 1: The Derivative of a Constant

$\dfrac{d}{dx}[c]=0,$ where *c* is a constant.

Example 3:

A. If $f(x)=-17,$ find $f'(x)$.

B. If $f(x)=\sqrt{11}$ find $f'(x)$.

Rule 2: The Power Rule

$\dfrac{d}{dx}\left[x^n\right]=nx^{n-1}$ for any real number *n*

Example 4: If $f(x)=x^5,$ find $f'(x)$.

Example 5: If $f(x)=\sqrt{x}$, find $f'(x)$.

Example 6: If $f(x)=\frac{1}{x^3}$, find $f'(x)$.

Rule 3: Derivative of a Constant Multiple of a Function

$\frac{d}{dx}\left[cf(x)\right]=c\frac{d}{dx}\left[f(x)\right]$ where c is any real number

Example 7: If $f(x)=-6x^4$, find $f'(x)$.

Example 8: If $f(x)=\frac{9}{5\sqrt[3]{x^{20}}}$, find $f'(x)$.

Rule 4: The Sum/Difference Rule

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

Example 9: Find the derivative: $f(x) = -8x^4 - \frac{2}{x^6} + \frac{7}{x} + 10x - 1.$

Let's revisit Example 1:

Example 10: Suppose the distance covered by a car can be measured by the function $s(t) = 4t^2 + 32t$, where $s(t)$ is given in feet and t is measured in seconds. Find the instantaneous velocity of the car when $t = 4$.

Recall that "instantaneous velocity" is the same as the derivative.

Note, there are many other rules for finding derivatives "by hand." We will not be using those in this course. Instead, we will use GeoGebra for finding more complicated derivatives.